

SOME VARIATIONAL PROBLEMS WITH ACCOUNT TAKEN OF THE COST OF CONTROL IN HILBERT SPACES

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Abstract

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MATHEMATICS

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SOME VARIATIONAL PROBLEMS WITH ACCOUNT TAKEN OF THE COST OF CONTROL IN HILBERT SPACES

1. Denote by H a real Hilbert space with scalar product $(\cdot, \cdot)$ and norm $\|\cdot\|$. The norms and scalar products of the Hilbert spaces H_1, H_2 , and H_3 will be denoted by the corresponding subscript. Let $\mathbf{L}_2 = \mathbf{L}_2(H, [0, T])$ be the Hilbert space of vector-functions with values in H (see ⁽¹⁾).

Theorem. Let $D(L_2 \rightarrow L_2)$, $F(H \rightarrow L_2)$, $B(L_2 \rightarrow H_3)$, and $C(H \rightarrow H_3)$ be linear bounded operators. Then for any prescribed pairs $\xi_1 \in H_3$, $\xi_2(t) \in L_2$ there exists a pair $\eta \in L_2$, $\zeta \in H$, minimizing the functional

$$E(\eta, \zeta) = \|B\eta + C\zeta - \xi_1\|_3^2 + \int_0^T \|D\eta + F\zeta - \xi_2\|^2 dt + \alpha \int_0^T \|\eta\|^2 dt + \gamma \|\zeta\|^2, \quad (1)$$

and η and ζ are determined uniquely from the system of equations

$$\begin{aligned} (B^*B + D^*D + \alpha)\eta + (B^*C + D^*F)\zeta &= B^*\xi_1 + D^*\xi_2, \\ (C^*B + F^*D)\eta + (C^*C + F^*F + \gamma)\zeta &= C^*\xi_1 + F^*\xi_2. \end{aligned} \quad (2)$$

Here and throughout the sequel $\alpha, \gamma > 0$.

For the proof of the theorem one uses the result from ⁽²⁾, if for H_1 we take $L_2 \times H$ with scalar product of the elements $x = \langle \eta, \zeta \rangle$ and $y = \langle \psi, \varphi \rangle$, defined by the formula $(x, y)_1 = \alpha[\eta, \psi] + \gamma(\zeta, \varphi)$, and in $H_2 = H_1 \times L_2$ introduce the scalar product of the elements $z = \langle \langle \xi, \eta \rangle \rangle$ and $r = \langle \langle \tau, \psi \rangle \rangle$ by the formula $(z, r)_2 = (\xi, \tau)_3 + [\eta, \psi]$, and as the operator $K(H_1 \rightarrow H_2)$ from ⁽²⁾ take the operator $Kx = \langle \langle B\eta + C\zeta, D\eta + F\zeta \rangle \rangle$.

Then

$$K^*z = \langle \alpha^{-1}(B^*\xi + D^*\eta), \gamma^{-1}(C^*\xi + F^*) \rangle.$$

2. We now consider the Cauchy problem for a differential equation in the Hilbert space H

$$u'(t) + Au(t) = f(t), \quad u(0) = \varphi, \quad 0 \leq t \leq T, \quad (3)$$

where A is a linear bounded self-adjoint operator in H .

In the paper ⁽²⁾ (see also ⁽³⁾) the problem was considered of finding a pair $\langle f, \varphi \rangle \in H_1$ minimizing the functionals

$$E_1(\langle f, \varphi \rangle) = \|u(T) - \psi\|^2 + \alpha \int_0^T \|f(t)\|^2 dt + \gamma \|\varphi\|^2,$$

$$E_2(\langle f, \varphi \rangle) = \int_0^T \|u(t) - g(t)\|^2 dt + \alpha \int_0^T \|f(t)\|^2 dt + \gamma \|\varphi\|^2,$$

where $\psi \in H$ and $g(t) \in L_2$ are prescribed elements, and $u(t)$ is the solution of the Cauchy problem (3) with unbounded self-adjoint linear operator A .

We now consider analogous problems, where the “costs of control” remain the same, while into the “cost of deviation” we introduce the derivative of the solution $u'(t)$. More precisely, the following is considered

Problem 1. Find a pair $\langle f, \varphi \rangle \in H_1$ minimizing the functional

$$E_1(\langle f, \varphi \rangle) = \|u(T) - \psi\|^2 + \int_0^T \|u'(t) - h(t)\|^2 dt + \alpha \int_0^T \|f(t)\|^2 dt + \gamma \|\varphi\|^2, \quad (4)$$

where $u(t)$ is the solution of problem (3), and $\psi \in H$ and $h(t) \in L_2$ are given elements. The problem has a solution, and a unique one, if in the theorem we put $H_3 = H$ and

$$Bf = \int_0^T e^{-(T-s)A} f(s) ds, \quad C\varphi = e^{-TA}\varphi,$$

$$Df = f(t) - A \int_0^t e^{-(t-\tau)A} f(\tau) d\tau, \quad F\varphi = -Ae^{-tA}\varphi. \quad (5)$$

It is not difficult, in particular, to establish that

$$D^* Df = f(t) - \frac{1}{2} A \int_0^T [e^{-(2T-t-\tau)A} + e^{-|t-\tau|A}] f(\tau) d\tau,$$

$$F^* Df = -\frac{1}{2} A \int_0^T [e^{-(2T-\tau)A} + e^{-\tau A}] f(\tau) d\tau,$$

$$D^* F\varphi = -\frac{1}{2} A (e^{-(2T-t)A} + e^{-tA}) \varphi, \quad F^* F\varphi = \frac{1}{2} A (I - e^{-2TA}) \varphi,$$

$$B^* Bf = \int_0^T e^{-(2T-t-\tau)A} f(\tau) d\tau, \quad C^* Bf = \int_0^T e^{-(2T-\tau)A} f(\tau) d\tau.$$

Then system (2), after some transformations, takes the form

$$\begin{aligned} (1 + \alpha)f(t) + \int_0^T W(t, \tau)f(\tau) d\tau + W(t, 0)\varphi &= G_1(t), \\ (A + \gamma I)\varphi + \int_0^T W(0, \tau)f(\tau) d\tau + W(0, 0)\varphi &= G_1(0) - h(0), \end{aligned} \quad (6)$$

where

$$\begin{aligned} W(t, \tau) &= \left(I - \frac{1}{2}A\right) e^{-(2T-t-\tau)A} - \frac{1}{2}A e^{-|t-\tau|A}, \\ G_1(t) &= h(t) + e^{-(T-t)A}\psi - A \int_0^T e^{-(\tau-t)A} f(\tau) d\tau. \end{aligned}$$

From (6), for $t = 0$, we obtain

$$\varphi = (A + \gamma I)^{-1} [(1 + \alpha)f(0) - h(0)]. \quad (7)$$

Therefore system (6) is replaced by the loaded integral equation

$$\begin{aligned} (1 + \alpha)f(t) + \int_0^T W(t, \tau)f(\tau) d\tau + (1 + \alpha)W(t, 0)(A + \gamma I)^{-1}f(0) &= \\ &= G_1(t) + W(t, 0)(A + \gamma I)^{-1}h(0). \end{aligned} \quad (8)$$

Problem 2. Let $h_1(t)$ and $h_2(t)$ be given functions from L_2 . Find a pair $\langle f, \varphi \rangle \in H_1$ minimizing the functional

$$E_2(\langle f, \varphi \rangle) = \int_0^T \|u(t) - h_1(t)\|^2 dt + \int_0^T \|u'(t) - h_2(t)\|^2 dt + \alpha \int_0^T \|f(t)\|^2 dt + \gamma \|\varphi\|^2, \quad (9)$$

where $u(t)$ is the solution of the Cauchy problem (3).

Functional (9), as in problem 1, is a special case of functional (1), if in the theorem we put $H_3 = L_2$, define the operators $D(L_2 \rightarrow L_2)$, $F(H \rightarrow L_2)$ as in (5), and define the operators $B(L_2 \rightarrow L_2)$ and $C(H \rightarrow L_2)$ as follows:

so that

$$Bf = \int_0^t e^{-(t-\tau)A} f(\tau) d\tau, \quad C\varphi = e^{-tA} \varphi.$$

Then system (2) takes the form

$$\begin{aligned} (1 + \alpha)f(t) + \int_t^T \int_0^s e^{-(2s-\tau-t)A} f(\tau) d\tau ds + \int_0^T \Psi(t, \tau) f(\tau) d\tau + \\ + \int_t^T e^{-(2\tau-t)A} \varphi d\tau + \Psi(t, 0)\varphi = G_2(t), \\ (A + \gamma I)\varphi + \int_0^T \int_0^s e^{-(2s-\tau)A} f(\tau) d\tau ds + \int_0^T \Psi(0, \tau) f(\tau) d\tau + \\ + \int_0^T e^{-2\tau A} \varphi d\tau + \Psi(0, 0)\varphi = G_2(0) - h_2(0), \end{aligned} \quad (10)$$

where

$$\Psi(t, \tau) = W(t, \tau) - e^{-(2T-t-\tau)A}, \quad G_2(t) = h_2(t) + \int_t^T e^{-(\tau-t)A} (h_1(\tau) - Ah_2(\tau)) d\tau.$$

This system is equivalent to the equation

$$\begin{aligned}
(1 + \alpha)f(t) + \int_t^T \int_0^s e^{-(2s-\tau-t)A} f(\tau) d\tau ds + \\
+ \int_0^T \Psi(t, \tau) f(\tau) d\tau + (1 + \alpha)(A + \gamma I)^{-1} \int_t^T e^{-(2\tau-t)A} f(0) d\tau + \\
+ (1 + \alpha)\Psi(t, 0)(A + \gamma I)^{-1} f(0) = \\
= G_2(t) + \Psi(t, 0)(A + \gamma I)^{-1} h_2(0) + (A + \gamma I)^{-1} \int_t^T e^{-(2\tau-t)A} A h_2(0) d\tau.
\end{aligned} \tag{11}$$

3. Let us present one of the possible realizations of these variational problems. Let $K(x, y) = R(x)R(y)$ be a continuous function for $0 \leq x, y \leq 1$, and

$$\int_0^1 R^2(\xi) d\xi = q < +\infty.$$

If, for each fixed t , for functions $u(t, x) \in L_2[0, 1]$ one defines the operator A by the formula

$$Au = (Au)(t, x) = \int_0^1 R(x)R(y)u(t, y) dy, \tag{*}$$

then A will be a linear bounded self-adjoint operator mapping $L_2[0, 1]$ into itself.

Let a certain process be described by the Cauchy problem for the integro-differential equation

$$\frac{\partial u(t, x)}{\partial t} + \int_0^1 R(x)R(y)u(t, y) dy = f(t, x), \quad u(0, x) = \varphi(x), \quad t \geq 0.$$

Here $H = L_2[0, 1]$, the operator A is defined by formula (*) and, for any $v \in H = L_2[0, 1]$,

$$e^{-tA}v = v + \frac{e^{-tq} - 1}{q}Av, \quad (A + \gamma I)^{-1}v = \frac{1}{\gamma}v - \frac{1}{\gamma(\gamma + q)}Av.$$

The functionals E_1 and E_2 in this case take the form

$$E_1(\langle f, \varphi \rangle) = \int_0^1 |u(T, x) - \psi(x)|^2 dx + \int_0^T \int_0^1 \left| \frac{\partial u(t, x)}{\partial t} - h(t, x) \right|^2 dx dt +$$

$$+ \alpha \int_0^T \int_0^1 |f(t, x)|^2 dx dt + \gamma \int_0^1 |\varphi(x)|^2 dx,$$

$$\begin{aligned} E_2(\langle f, \varphi \rangle) &= \int_0^T \int_0^1 |u(t, x) - h_1(t, x)|^2 dx dt + \int_0^T \int_0^1 \left| \frac{\partial u(t, x)}{\partial t} - h_2(t, x) \right|^2 dx dt \\ &+ \alpha \int_0^T \int_0^1 |f(t, x)|^2 dx dt + \gamma \int_0^1 |\varphi(x)|^2 dx. \end{aligned}$$

Problem 1 becomes the problem of finding an optimal pair $\langle f, \varphi \rangle$ that realizes the best (in a certain sense) approximation of the final result $u(T, x)$ of the process $u(t, x)$ to a prescribed function $\psi(x)$. Its solution is found from the equations

$$\begin{aligned} (1 + \alpha)f(t, x) &+ \int_0^T \int_0^1 \Phi(t, x, \tau, y, \tau)f(\tau, y) dy d\tau \\ &+ (1 + \alpha) \int_0^1 P(t, x, y)f(0, y) dy \\ &+ \int_0^T \left\{ \varepsilon(t, \tau)f(\tau, x) + \frac{1 + \alpha}{\gamma T} \varepsilon(t, 0)f(0, x) \right\} d\tau = G(t, x), \end{aligned} \quad (12)$$

$$\varphi(x) = \frac{1}{\gamma} [(1 + \alpha)f(0, x) - h(0, x)] - \frac{1}{\gamma(\gamma + q)} \int_0^1 [(1 + \alpha)f(0, y) - h(0, y)] R(x) R(y) dy, \quad (13)$$

where

$$\Phi(t, x, y, \tau) = [U(t, \tau) - 1/q] R(x) R(y),$$

$$U(t, \tau) = (2 - q)e^{-(2T-t-\tau)q}/2q - e^{-|t-\tau|q}/2,$$

$$P(t, x, y) = [U(t, 0)/(\gamma + q) - 1/\gamma q] R(x) R(y), \quad \varepsilon(t, \tau) = \varepsilon(t, 0) = 1,$$

$$\begin{aligned} G(t, x) &= \int_0^1 P(t, x, y)h(0, y) dy + \frac{1}{q} \int_0^1 (e^{-(T-t)q} - 1) R(x) R(y) \psi(y) dy \\ &- \int_t^T \int_0^1 e^{-(\tau-t)q} R(x) R(y) h(y, \tau) dy d\tau + h(t, x) + h(0, x)/\gamma + \psi(x). \end{aligned}$$

Problem 2 becomes the problem of finding an optimal pair $\langle f, \varphi \rangle$ that realizes the best (in a certain sense) approximation of the process $u(t, x)$ to the prescribed process $h_1(t, x)$ over a certain time interval $(0, T)$. Its solution $f(x, t)$ is found from (12), (13), where

$$\Phi(t, x, \tau, y) = [Q(t, \tau) - \omega(t, \tau)/q]R(x)R(y), \quad \omega(t, \tau) = \begin{cases} T - t, & 0 \leq \tau \leq t, \\ T - \tau, & t \leq \tau \leq T, \end{cases}$$

$$Q(t, \tau) = [(1 - q^2)e^{-|t-\tau|q} - (1 + q^2)e^{-(2T-t-\tau)q}]/2q^2, \quad \varepsilon(t, \tau) = \omega(t, \tau),$$

$$P(t, x, y) = [Q(t, 0)/(\gamma + q) - \omega(t, 0)/\gamma q]R(x)R(y),$$

$$G(t, x) = \int_t^T \int_0^1 \left[\frac{e^{-(\tau-t)q} - 1}{q} h_1(\tau, y) - e^{-(\tau-t)q} h_2(\tau, y) \right] R(x)R(y) dy d\tau + \\ + \int_t^T h_1(\tau, x) d\tau + \int_0^1 P(t, x, y) h_2(0, y) dy + h_2(t, x) + \frac{\omega(t, 0)}{\gamma} h_2(0, x).$$

Other problems of this type, concerning problems of best approximation to a prescribed temperature regime, are contained in (4-7).

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