

ON PROPERTIES OF BOUNDED SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS IN $\backslash(B\backslash)$ -SPACES

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1968

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Abstract

Full Text

UDC 517.919.2+517.948

MATHEMATICS

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ON PROPERTIES OF BOUNDED SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS IN B -SPACES

(Presented by Academician V. I. Smirnov on 27 III 1967)

1. Let Z be a complex B -space, and let A be some endomorphism* of it with spectrum not intersecting the imaginary axis. We shall say that a set Ω , whose elements are functions defined on $(-\infty, +\infty)$ with values in Z , has property C if the equation

$$\dot{x} + Ax = \varphi(t) \quad (1)$$

has in Ω a unique solution for every $\varphi(t) \in \Omega$.**

It is known ⁽¹⁾ that the property C is possessed by the set of functions continuous and bounded on $(-\infty, +\infty)$, the set of continuous periodic functions, and the set of almost periodic functions. It seems important to indicate a criterion allowing one to judge whether a given set Ω possesses property C .

In the present paper this problem is solved for one family of sets Ω : conditions are indicated that are necessary and sufficient in order that a set Ω from this family possess property C . Some sufficient conditions are also indicated. The results of the monograph ⁽¹⁾ listed above follow from the theorems of this paper.

2. We introduce notation.

Φ is the set of all strongly measurable and almost everywhere bounded functions with values in Z , defined on $(-\infty, +\infty)$.

ν is a functional defined on Φ (the value $+\infty$ is also allowed among its values) such that for any φ and ψ from Φ and any complex ξ the following hold: 1) $\nu(\varphi) \geq 0$; 2) $\nu(\varphi + \psi) \leq \nu(\varphi) + \nu(\psi)$; 3) $\nu(\xi\varphi) = |\xi|\nu(\varphi)$; 4***) $\nu[\varphi(t + \tau)] = \nu[\varphi(t)]$ for any $\tau \in (-\infty, +\infty)$; 5) from the inequality $\|\varphi(t)\| \leq \|\psi(t)\|$, holding on $(-\infty, +\infty)$, follows the inequality $\nu(\varphi) \leq \nu(\psi)$.

Φ_ν is the subset of Φ formed by all elements of Φ satisfying the inequality $\nu(\varphi) < +\infty$. It is obvious that Φ_ν is a complex linear system.

We shall call φ and ψ from Φ ν -equivalent if $\nu(\varphi - \psi) = 0$. It is obvious that Φ_ν can be represented as a sum of nonintersecting classes of elements pairwise ν -equivalent; the functional ν is constant on each such class.

C_ν is the normed linear space whose elements are the above-mentioned classes of elements of Φ_ν , and the norm of an element is taken to be the value of the functional ν on such a class. It is not difficult to see that all spaces $L_p(-\infty, +\infty)$, $1 \leq p \leq +\infty$, are spaces of type C_ν .

* By an endomorphism of a normed linear space this paper means a linear bounded operator mapping this space into itself.

** A solution of the differential equation (1) is understood to be a continuous function satisfying this equation almost everywhere on $(-\infty, +\infty)$.

*** In connection with the use of the shift operation, the paper allows a double notation: sometimes we write $\varphi \in \Phi$, and sometimes $\varphi(t) \in \Phi$.

$E(X)$ is the set of endomorphisms of the linear normed space X . It is obvious that $E(Z) \subset E(C_\nu)$.

P_τ is the shift operator, i.e., the operator defined on Φ by the equality $P_\tau[\varphi(t)] = \varphi(t + \tau)$; P is the set of all P_τ , $\tau \in (-\infty, +\infty)$.

S_h is the operator defined on Φ by the equality*

$$S_h[\varphi(t)] = \frac{1}{h} \int_t^{t+h} \varphi(\tau) d\tau;$$

S is the set of all S_h , $h \neq 0$, $h \in (-\infty, +\infty)$. It is easy to see that $P \subset E(C_\nu)$ and $S \subset E(C_\nu)$.

3. For the problem studied in this paper, the space C_ν turns out to be a rather natural object, as is seen from the following theorem.

Theorem 1. *If $\varphi(t) \in C_\nu$, then equation (1) has a unique solution in C_ν .*

In Theorem 1, the spaces $L_p(-\infty, +\infty)$, $1 \leq p \leq +\infty$, may be taken as C_ν . For $p = +\infty$ one obtains the previously known result on the existence of a unique bounded solution of equation (1).

Despite its generality, Theorem 1 gives only one-sided information about the bounded solution of equation (1). To study more subtle properties of this solution, a new type of space is introduced.

Definition. Let $\tilde{\Phi}$ be a complete subspace of the space C_ν . We shall call $\tilde{\Phi}$ a B -space if $E(Z) \subset E(\tilde{\Phi})$.

Theorem 2. *Let $\tilde{\Phi}$ be a B_ν -space. In order that, for every $\varphi(t) \in \tilde{\Phi}$ and every A^{**} , equation (1) have a unique solution in $\tilde{\Phi}$, it is necessary and sufficient that $S \subset E(\tilde{\Phi})$.*

Example. Let $\tilde{\Phi}$ be the set of all continuous bounded functions defined on $(-\infty, +\infty)$ and having the limit $\lim_{t \rightarrow +\infty} \varphi(t)$. If ν is introduced by the equality

$$\nu(\varphi) = \sup_{-\infty < t < +\infty} \|\varphi(t)\|,$$

then $\tilde{\Phi}$ will be a B_ν -space. For every $\varphi(t) \in \tilde{\Phi}$ and $h \neq 0$, the function

$$\tilde{\varphi}(h, t) = \frac{1}{h} \int_t^{t+h} \varphi(\tau) d\tau$$

will be continuous; the limit will also exist:

$$\lim_{t \rightarrow +\infty} \tilde{\varphi}(h, t) = \lim_{t \rightarrow +\infty} \varphi(t).$$

Consequently, $\tilde{\varphi}(t, h) \in \tilde{\Phi}$, and the space $\tilde{\Phi}$ introduced by us satisfies all the conditions of Theorem 2.

The conditions of Theorem 2 are checked just as simply for a broad class of spaces. In particular, these conditions are satisfied by the B_ν -spaces of T -periodic functions

$$\left(\nu(\varphi) = \lim_{M \rightarrow \infty} \left(\frac{1}{2M} \int_{-M}^{+M} \|\varphi(t)\|^p \right)^{1/p}, \quad 1 \leq p \leq +\infty \right);$$

uniformly almost periodic functions

$$\left(\nu(\varphi) = \sup_{-\infty \leq t < +\infty} \|\varphi(t)\| \right);$$

continuous functions, for each of which there exist two limits

$$\lim_{t \rightarrow -\infty} \varphi(t) \quad \text{and} \quad \lim_{t \rightarrow +\infty} \varphi(t)$$

$$\left(\nu(\varphi) = \sup_{-\infty < t < +\infty} \|\varphi(t)\| \right),$$

and many others.

Despite the convenience of using Theorem 2 in many concrete cases, it seems expedient to indicate some sufficient conditions that would imply the fulfillment of the condition $S \subset E(\tilde{\Phi})$, but not contain

* Here and below the integral is understood in the sense of Bochner.

** In accordance with the notation introduced earlier, this should be understood as follows: A is any element of the set of all endomorphisms of Z whose spectra do not intersect the imaginary axis.

would allow passage to the limit in the sense of the topology of the space Z . To obtain such results, however, one has to narrow the class of spaces under consideration.

Let us introduce the notation used in Theorems 3 and 3'.

Let $\tilde{\Phi}$ be a B_ν -space. We shall call $\tilde{\Phi}$ a $\tilde{B}_\nu$ -space if the following conditions are satisfied: a) every function that differs from zero only on a set of measure 0 is ν -equivalent to 0; b) convergence of a sequence $\{\varphi_n(t)\}$ in the space $\tilde{\Phi}$ implies its convergence in measure on $(-\infty, +\infty)$.

Let $\tilde{\Phi}$ be such a B_ν -space that $P \subset E(\tilde{\Phi})$. We shall denote by $\tilde{\Phi}_\tau = \{\varphi_\tau\}$ the set of all functions of the argument τ , with values in $\tilde{\Phi}$, of the form $\varphi_\tau = \varphi(t + \tau)$, where $\varphi(t) \in \tilde{\Phi}$.

Theorem 3. *Let $\tilde{\Phi}$ be a $\tilde{B}_\nu$ -space such that $P \subset E(\tilde{\Phi})$. If there exists an $h > 0$ such that all elements of the set $\tilde{\Phi}_\tau$ are strongly measurable on $[-h, +h]$, then $S \subset E(\tilde{\Phi})$.*

Theorem 3 contains a requirement of measurability of functions from $\tilde{\Phi}_\tau$ that is difficult to verify; more convenient for applications is its corollary,

Theorem 3'. *If $\tilde{\Phi}$ is such a $\tilde{B}_\nu$ -space that $P \subset E(\tilde{\Phi})$ and every function φ_τ from $\tilde{\Phi}_\tau$ is continuous at the point $\tau = 0$, then $S \subset E(\tilde{\Phi})$.*

We note that there exist $\tilde{B}_\nu$ -spaces satisfying the conditions of Theorem 2 but not satisfying the conditions of Theorems 3 and 3'. Such a space, in particular, is the space $\tilde{\Phi}$ from the example to Theorem 2; the B_ν -spaces of periodic and almost periodic functions also satisfy the conditions of Theorem 3'.

In conclusion, we note that the requirement of boundedness of the operator A can be weakened; it is required only that it belong to a certain class of closed operators.

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Received
22 III 1967

REFERENCES

1. M. G. Krein, *Lectures on the Theory of Stability of Solutions of Differential Equations in Banach Space*, Kiev, 1964.

Note: Figure translations are in progress. See original paper for figures.

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