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Abstract

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MATHEMATICS

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ON THE DIMENSION OF PRODUCTS OF BICOMPACTA

(Presented by Academician P. S. Aleksandrov on 27 II 1968)

§ 1. In the present section only finite covers and only normal spaces are considered. By the dimension of a space X we mean the large inductive dimension, denoted by $\text{Ind } X$. Let us recall the definition of this dimension: $\text{Ind } X \leq n$ if, for every closed set $F \subseteq X$ and every neighborhood OF of it, there exists a neighborhood V of the set F such that $V \subseteq OF$ and $\text{Ind}([V] \setminus V) \leq n - 1$. By definition $\text{Ind } X = -1$ if the space X is empty.

Definition 1. For a space X , a closed set F_1 is called a **boundary** if F_1 is the boundary of a set open in X .

Definition 2. A space X satisfies the condition of extension of refinements*, if for any closed sets F_1 and F_2 , $F_1 \supseteq F_2$, $\text{Ind } F_1 > \text{Ind } F_2$, F_2 a boundary in F_1 , the following holds: for every open cover ω of the set F_1 and every refinement α of the set F_2 , inscribed in the cover $\omega \wedge F_2^{**}$ with $\text{Ind gr } \alpha \leq \text{Ind } F_2 - 1$, there exists a refinement β of the set F_1 , inscribed in the cover ω , such that $\beta \wedge F_2 = \alpha$, $\text{gr } \beta \cap F_2 = \text{gr } \alpha$, and $\text{Ind gr } \beta \leq \text{Ind } F_2 - 1$.

Definition 3. We shall say that a bicom pactum X **belongs to the class of bicom pacta** if the following conditions are fulfilled for X :

I (weakened sum theorem). For any two closed sets F_1 and F_2 that are boundaries,

$$\text{Ind}(F_1 \cup F_2) = \max(\text{Ind } F_1, \text{Ind } F_2).$$

II. The bicom pactum X satisfies the condition of extension of refinements.

It turns out that for bicom pacta X belonging to the class , the following is true.

Theorem 1. Let

$$P = \prod_{i=1}^s X_i,$$

where $X_i \in (i = 1, 2, \dots, s)$. Then

$$\text{Ind } P \leq \text{Ind } X_1 + \dots + \text{Ind } X_s.$$

It is not hard to show that the class of bicompecta includes the following classes of bicompecta: a) all bicompecta whose large inductive dimension does not exceed one; b) all bicompecta that are Dowker spaces*** (see (2)), and consequently also all perfectly normal bicompecta.

The proof of part a) is contained in (4).

Corollary 1. Let $P = X \times Y$, where X is a one-dimensional bicompectum and Y is a perfectly normal bicompectum for which $\dim Y = \text{ind } Y = \text{Ind } Y = n$. Then $\dim P = \text{ind } P = \text{Ind } P = n + 1$.

Proof. $\dim P \geq n + 1$ follows from (5), and $\text{Ind } P \leq n + 1$ follows from Theorem 1.

* A cover $\alpha = \{[O_i]\}$ ($i = 1, \dots, n$) is called a **refinement** if its elements are the closures of pairwise disjoint open sets O_i , called the kernels of the elements of the refinement,

$$\text{gr } \alpha = \bigcup_{j=1}^n [O_i] \setminus O_i.$$

** Let $F \subseteq X$: 1) let $\omega = \{u_1, \dots, u_s\}$ be an open cover of the space X ; then $\omega \wedge F = \{F \cap u_1, \dots, F \cap u_s\}$; 2) let $\alpha = \{[O_1], \dots, [O_s]\}$ be a refinement of X ; then $\alpha \wedge F = \{[O_1 \cap F], \dots, [O_n \cap F]\}$.

*** A hereditarily normal space X is called a **Dowker space** if every open set U is covered by a point-finite system of open sets F_σ in X .

The proof of Theorem 1 proceeds as follows:

- 1) It is shown that in any open covering ω of the bicompectum P one can inscribe a refinement α^1 such that $\text{bd } \alpha^1$ is the sum of a finite number of summands of the form $\prod_{i=1}^s F_i$, where F_i is the boundary of some order in X_i , and

$$\sum_{i=1}^s \text{Ind } F_i = \sum_{i=1}^s \text{Ind } X_i - 1$$

for each summand of the set $\text{bd } \alpha^1$.

- 2) Then it is shown that in any open covering ω^1 of the set $\text{bd } \alpha^1$ one can inscribe a refinement α^2 of the set $\text{bd } \alpha^1$ such that $\text{bd } \alpha^2$ (relative to $\text{bd } \alpha^1$)

is the sum of a finite number of closed summands of the form $\prod_{i=1}^s F'_i$, and, for each summand of the set $\text{bd } \alpha^2$, we have

$$\sum_{i=1}^s \text{Ind } F'_i = \sum_{i=1}^s \text{Ind } X_i - 2 \quad (F'_i \subseteq X'_i).$$

After this procedure has been applied $\sum_{i=1}^s \text{Ind } X_i = l$ times, we obtain that $\text{bd } \alpha^l$ is the sum of a finite number of closed summands of the form $\prod_{i=1}^s F_i$, and

$$\sum_{i=1}^s \text{Ind } F_i = 0.$$

Consequently, the dimension of each set F_i is zero; therefore the dimension of each summand of the set $\text{bd } \alpha^l$ and the dimension of the set $\text{bd } \alpha^l$ itself is zero. Then, going back, we obtain that

$$\text{Ind } \text{bd } \alpha \leq \sum_{i=1}^s \text{Ind } X_i - 1$$

and, consequently,

$$\text{Ind } P \leq \sum_{i=1}^s \text{Ind } X_i.$$

Definition 4. We shall say that a bicom pactum X **belongs to the class of bicom pacta** A if X is a closed subset of some bicom pactum P which is the topological product of a finite number of bicom pacta from the class .

Remark 1. The class A is closed with respect to topological product.

Remark 2. Let $B = \prod_{i=1}^k A_i$, where $A_i \in A$ ($i = 1, \dots, k$) and each bicom pactum A_i is finite-dimensional. Then the bicom pactum B is also finite-dimensional.

§ 2. In this section we consider the question of the dimension of the product of an infinite number of arbitrary bicom pacta X with $\dim X \geq 1$.

The following notion of a closed partition between disjoint closed sets is known. Let C_1 and C_2 be closed sets of a space X , $C_1 \cap C_2 = \emptyset$. Then a closed set B is called a **partition** between C_1 and C_2 if $X \setminus B = V^1 \cup V^2$, $V^1 \cap V^2 = \emptyset$, the sets V^1, V^2 are open, and $C_1 \subseteq V^1$, $C_2 \subseteq V^2$.

P. S. Aleksandrov gave the following definition: a space X is **strongly infinite-dimensional** if there exists a sequence of pairs (C_i^1, C_i^2) , $C_i^1 \cap C_i^2 = \emptyset$ ($i =$

$1, 2, \dots$), of closed sets such that the intersection of any closed sets B_i ($i = 1, 2, \dots$), each of which is a partition for the corresponding pair (C_i^1, C_i^2) , is nonempty. If the space X is strongly infinite-dimensional in the sense of P. S. Aleksandrov, then we shall say that it is A -strongly infinite-dimensional.

Now we introduce a strengthening of the notion of partition.

Definition 5. Let there be given closed sets C^1 and C^2 in X , with $C^1 \cap C^2 = \emptyset$. A set B is called a **thick partition** between C^1 and C^2 if $X \setminus \text{Int } B = \Phi_1 \cup \Phi_2$, $\Phi_1 \cap \Phi_2 = \emptyset$, the sets Φ_1, Φ_2 are closed, $C^i \subseteq \text{Int } \Phi_i$ ($i = 1, 2$) ($\text{Int } B$ denotes the interior of the set B), and $B \cap (C^1 \cup C^2) = \emptyset$.

If the space X is normal, then the following assertions are equivalent:

- I. In the space X there exist n pairs of closed sets (C_i^1, C_i^2) ($i = 1, 2, \dots, n$), $C_i^1 \cap C_i^2 = \emptyset$, such that the intersection of any closed sets B_i ($i = 1, 2, \dots, n$), each of which is a thick partition for the pair (C_i^1, C_i^2) , is nonempty.
- II. In the space X there exist n pairs of closed sets (C_i^1, C_i^2) ($i = 1, 2, \dots, n$), $C_i^1 \cap C_i^2 = \emptyset$, such that the intersection of arbitrary closed sets B_i ($i = 1, 2, \dots, n$), each of which is a partition for the pair (C_i^1, C_i^2) , is nonempty.

Definition 6. A normal space X will be called **L -strongly infinite-dimensional** if there exists a sequence of pairs (C_i^1, C_i^2) ($i = 1, 2, \dots$), $C_i^1 \cap C_i^2 = \emptyset$, of closed sets such that the intersection of arbitrary closed sets B_i ($i = 1, 2, \dots$), each of which is a thick partition for the corresponding pair (C_i^1, C_i^2) , is nonempty.

Lemma 1. *If the space X is a bicompactum, then the definitions of L -strong infinite-dimensionality and A -strong infinite-dimensionality are equivalent.*

Proof. a) From A -strong infinite-dimensionality, obviously, L -strong infinite-dimensionality follows.

- b) Let the bicompactum X be L -strongly infinite-dimensional. Suppose that it is not A -strongly infinite-dimensional. Then for any sequence of pairs of closed sets (F_i^1, F_i^2) , $F_i^1 \cap F_i^2 = \emptyset$ ($i = 1, 2, \dots$), there exists a sequence of closed sets A_i ($i = 1, 2, \dots$) (each of which is a partition respectively for the pair (F_i^1, F_i^2)) such that

$$\bigcap_{i=1}^{\infty} A_i = \emptyset.$$

Consequently, also for the sequence of pairs of closed sets (C_i^1, C_i^2) participating in the definition of L -strong infinite-dimensionality, there exists a corresponding sequence of closed sets A_i ($i = 1, 2, \dots$). Since X is a bicompactum and

$$\bigcap_{i=1}^{\infty} A_i = \emptyset,$$

there exists an integer k such that

$$\bigcap_{i=1}^k A_i = \emptyset.$$

Therefore one can take open sets O_i ($i = 1, 2, \dots, k$) such that $A_i \subseteq O_i$, $[O_i] \cap (C_i^1 \cup C_i^2) = \emptyset$, and

$$\bigcap_{i=1}^k [O_i] = \emptyset.$$

Now, taking for each $j > k$ an arbitrary open set O_j such that $A_j \subseteq O_j$ and $[O_j] \cap (C_j^1 \cap C_j^2) = \emptyset$, we obtain such a sequence of closed sets $[O_i]$ ($i = 1, 2, \dots$) (which are thick partitions for the corresponding pairs (C_i^1, C_i^2)) that

$$\bigcap_{i=1}^{\infty} [O_i] = \emptyset.$$

We have arrived at a contradiction with the choice of the sequence of pairs of closed sets (C_i^1, C_i^2) ($i = 1, 2, \dots$).

Remark 3. Lemma 1 is also true for countably paracompact spaces (spaces in every countable open cover of which one can inscribe a locally finite cover (see (6))).

Theorem 2. Let $P = \prod_{\alpha \in A} X_\alpha$, where X_α are bicompacta, $\dim X_\alpha \geq 1$ for every $\alpha \in A$, and $m(A) \geq \aleph_0$. Then the bicompactum P is L -strongly infinite-dimensional.

Proof. It is enough to consider the case when

$$P = \prod_{i=1}^{\infty} X_i$$

is the topological product of a countable number of bicompacta X_i . Since $\dim X_i \geq 1$, in the bicompactum X_i there exist points $x_i^1 \neq x_i^2$ which cannot be separated by the empty set. Take the sets

$$C_i^1 = \prod_{j \neq i} X_j \times x_i^1, \quad C_i^2 = \prod_{j \neq i} X_j \times x_i^2.$$

Then C_i^1 and C_i^2 are closed in the bicompactum P , $C_i^1 \cap C_i^2 = \emptyset$, and it remains only to prove that the intersection of arbitrary closed sets B_i ($i = 1, 2, \dots$), each of which

is a thick partition for the pair (C_i^1, C_i^2) , respectively, is nonempty. To prove the last assertion, we note that for this it is sufficient to prove the nonemptiness of the intersection of any finite number of the sets B_i , since then the assertion we need will follow from the bicomcompactness of the space P . Thus, the proof of Theorem 2 has been reduced to the proof of the following assertion.

Let

$$P_n = \prod_{i=1}^n X_i,$$

where the X_i are bicompacts and $\dim X_i \geq 1$ for every $i = 1, 2, \dots, n$. Then the sequence of pairs of closed sets (C_i^1, C_i^2) (where

$$C_i^1 = \prod_{j \neq i}^n X_j \times x_i^1, \quad C_i^2 = \prod_{j \neq i}^n X_j \times x_i^2$$

) is such that the intersection of arbitrary closed sets B_i ($i = 1, 2, \dots, n$), each of which is a thick partition for the pair (C_i^1, C_i^2) , is nonempty.

We shall prove the last assertion for the case $n = 2$. Let

$$P_2 = \prod_{i=1}^2 X_i, \quad \dim X_i \geq 1 \quad (i = 1, 2),$$

and let B_1, B_2 be thick partitions for the pairs of closed sets (C_1^1, C_1^2) and (C_2^1, C_2^2) , respectively. Suppose that $B_1 \cap B_2 = \emptyset$. Then the sets

$$F_i^1 = B_i \cap \Phi_i^1, \quad F_i^2 = B_i \cap \Phi_i^2$$

(where $\Phi_i^1 \cup \Phi_i^2 = P_2 \setminus \text{Int } B_i$ ($i = 1, 2$) and $\Phi_i^1 \cap \Phi_i^2 = \emptyset$) are pairwise disjoint, i.e.

$$F_i^1 \cap F_i^2 = \emptyset$$

for $i = 1, 2$. For fixed i , the sets $F_i^1, F_i^2, C_i^1, C_i^2$ are also pairwise disjoint. Therefore one can take an open cover ω of the bicomcompact P_2 such that:

$$(*)$$

no element of this cover intersects simultaneously an arbitrary pair of disjoint sets from the system

$$F_i^1, F_i^2, B_i, \Phi_i^1, \Phi_i^2, C_i^1, C_i^2 \quad (i = 1, 2).$$

We refine this cover ω by an open cover

$$\tilde{\omega} = \{\omega_1 \times \omega_2\},$$

whose elements are sets of the form $(u_i^1 \times u_j^2)$, where $u_i^1 \in \omega_1, u_j^2 \in \omega_2$, and ω_1, ω_2 are open covers of the bicompacts X_1, X_2 , respectively. Let $\pi_i : P_2 \rightarrow X_i$ be

the natural projection of P_2 onto the factor X_i . Then one may require that no element of the cover ω_i intersect a pair of sets $(x_i^1, \pi_i(B_i))$ and $(x_i^2, \pi_i(B_i))$. Now take, for each ω_i , an ω_i -map f into the body of the nerve $\widetilde{K}_i$ of the cover (see (2)). Then the points $f_i(x_i^1)$ and $f_i(x_i^2)$ lie in one component of the polyhedron $\widetilde{K}_i$, and therefore there exists a segment I_i whose initial point is $f_i(x_i^1)$ and whose endpoint is $f_i(x_i^2)$. Let

$$X_i^+ = f_i^{-1}(I_i).$$

The set X_i^+ is closed in X_i , and $x_i^j \in X_i^+$ ($j = 1, 2$). The mapping

$$f = f_1 \times f_2$$

of the bicomact P_2 maps the set

$$P_2^+ = X_1^+ \times X_2^+$$

onto a square, and the sets

$$P_2^+ \cap C_i^j = +C_i^j = x_i^j \times X_k^+ \quad (i = 1, 2; i = k; k = 1, 2; j = 1, 2)$$

are mapped onto opposite sides of the square for fixed i . Let

$$\omega^+ = P_2^+ \wedge \omega;$$

then

$$f^+ = f/P_2^+,$$

the restriction of the mapping f to the set P_2^+ , will be an ω^+ -map, and

$$B_i^+ = B_i \cap P_2^+.$$

The set $f^+(B_i^+)$ separates the sets $f^+(+C_i^1)$ and $f^+(+C_i^2)$, since f^+ is an ω^+ -map and the cover ω of the bicomact P_2 satisfies condition (*). Consequently,

$$f^+(B_1^+) \cap f^+(B_2^+) = \emptyset,$$

and this contradicts the fact that

$$B_1^+ \cap B_2^+ = \emptyset,$$

f^+ is an ω^+ -map, and no element of the cover ω^+ intersects simultaneously the sets B_1^+ and B_2^+ . Consequently,

$$B_1^+ \cap B_2^+ = P_2^+ \cap (B_1 \cap B_2) \neq \emptyset,$$

i.e.

$$B_1 \cap B_2 \neq \emptyset.$$

Theorem 2 is proved.

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