

FACTORIZATION OF MAPPINGS INTO METRIC SPACES

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Abstract

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MATHEMATICS

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FACTORIZATION OF MAPPINGS INTO METRIC SPACES

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All spaces under consideration are assumed to be normal. For a finite space X we set $w(X) = \aleph_0$. The first factorization theorem for a mapping of a bicomactum onto a bicomactum was obtained by S. Mardešić in ⁽¹⁾; the first factorization theorem for a mapping of a normal space into a metric one was obtained by the author in ⁽²⁻⁴⁾; for further results see ^(5,6). Mardešić's factorization theorems from ⁽¹⁾ and mine from ^(3,4) are definitive in the sense that in them factorization is carried out both with respect to dimension and with respect to weight. At the same time, in the papers ^(5,6) factorization is carried out mainly with respect to dimension (it is not clear whether one may assume, for example, $w(hA_\lambda) \leq w(fA_\lambda)$ in Theorem 2 of ⁽⁵⁾, or $w(hF_i) \leq w(fF_i)$ in Theorem 3 of ⁽⁶⁾*). It turns out that in the case of mappings into metric spaces complete factorization (both with respect to weight and with respect to dimension) is possible.

Theorem 1. a) If the system π of closed sets A_α , $\alpha \in \mathfrak{A}$, of a space X is σ -locally finite, then for every mapping f of the space X into a metric space R there exist a metric space S and mappings $g : X \rightarrow S$ and $h : S \rightarrow R$ such that: 1) $f = hg$, and the mapping h is perfect; 2) $w(B) \leq w(hB)$ for every set $B \subset S$; 3) $\dim S \leq \dim X$, $\dim[gA_\alpha] \leq \dim A_\alpha$ for every $\alpha \in \mathfrak{A}$.

b) *If the system π of sets A_α , $\alpha \in \mathfrak{A}$, having type F_σ in X , is σ -strongly locally countable**, then for every mapping f of the space X into a metric space R there exist a metric space S and mappings $g : X \rightarrow S$ and $h : S \rightarrow R$ such that assertions 1) and 2) of part a) hold; in addition: 3) $\dim S \leq \dim X$, and for every $\alpha \in \mathfrak{A}$ the set gA_α is contained in a set B_α of type F_σ in S with $\dim B_\alpha \leq \dim A_\alpha$.*

If in part b) the space X is paracompact, then the condition on the system π can be weakened: it is enough to assume that it is σ -locally countable***.

Basic definitions. A system π of normal subsets A_α , $\alpha \in \mathfrak{A}$, of a space X will be called: a) **weakly factorizable**,

* It is likewise not clear whether $\dim[hF_i] \leq \dim F_i$.

** We shall call a system π of subsets of a space X strongly locally countable if there exists a locally finite open cover of the space X , each element of which intersects no more than a countable set of elements of the system π . A system that can be decomposed into the sum of a countable number of strongly locally countable subsystems will be called σ -strongly locally countable.

*** The formulated Theorem 1 is a more general and stronger assertion than Theorem 3 from the paper ⁽⁶⁾ of A. V. Arhangel'skii. Incidentally, we note that the derivation given by him in ⁽⁶⁾ of Theorem 6 from Theorem 3 of that paper is not convincing. For the derivation to be correct, in Theorem 3 of ⁽⁶⁾ it is necessary additionally to require the validity of the relations $\dim[hF_i] \leq \dim F_i$, which is formulated in Theorem 1 above, in particular, and is achieved. I note further that neither in ⁽²⁾ nor in ^(3,4) did I use spectral technique in proving the factorization theorem (see the remark in ⁽⁶⁾).

b) **factorizable**, c) **strongly factorizable**, d) **strongly* factorizable** with respect to a mapping $f : X \rightarrow Y$, if there exist a space Z and mappings $g : X \rightarrow Z$ and $h : Z \rightarrow Y$ such that the following relations hold: 1) $f = hg$; 2) $w(gA_\alpha) \leq w(fA_\alpha)$ for every $\alpha \in \mathfrak{A}$; 3) respectively: a) $\dim g\tilde{A}_\alpha^* \leq \dim A_\alpha$, $\alpha \in \mathfrak{A}$; b) $\dim[gA_\alpha]_Z \leq \dim A_\alpha$, $\alpha \in \mathfrak{A}$; c) the mapping h is perfect** and $\dim[gA_\alpha]_Z \leq \dim A_\alpha$, $\alpha \in \mathfrak{A}$; d) the mapping h is perfect and for every $\alpha \in \mathfrak{A}$ the set gA_α is contained in a set B_α of type F_σ in Z with $\dim B_\alpha \leq \dim A_\alpha$. If $w(B) \leq w(hB)$ for every set $B \subseteq Z$, then we shall say that the system π is **factorizable without narrowing**. If a system π of subsets of the space X is (strongly, strongly*, weakly) factorizable (without narrowing) with respect to every mapping of the space X into any space Y from the class of spaces $\mathfrak{M}$ in such a way that $Z \in \mathfrak{M}$, then the system π will be called ***(strongly, strongly*, weakly) factorizable** (without narrowing)** with respect to the class of spaces $\mathfrak{M}$.

In what follows we shall be interested mainly in the factorization of mappings into metrizable spaces and into spaces which in a certain sense are close to them.

I. It turns out that the factorization of mappings into metrizable spaces reduces to the case of mappings into spaces with a countable base; namely, the following is true.

Reduction Lemma 1. *If a system π of subsets A_α , $\alpha \in \mathfrak{A}$, of a space X : a) is weakly factorizable, b) factorizable, c) strongly factorizable, or d) strongly* factorizable with respect to the class $\mathfrak{M}_c$ of spaces with a countable base, then the system π is factorizable without narrowing in the corresponding sense (a), b), c), or d)) with respect to the class $\mathfrak{M}_p$ of metric spaces.*

Definition. A property Q of a system of sets of a space X will be called **(strongly, strongly*, weakly) factorizable (without narrowing)**, respectively **(strongly, strongly*, weakly) τ -factorizable (without narrowing)**

with respect to a class of spaces $\mathfrak{M}$, if every system of sets of the space X possessing the property Q , respectively every system of sets of the space X decomposable into the sum of τ subsystems possessing the property Q , is (strongly, strongly*, weakly) factorizable (without narrowing) with respect to every mapping of the space X into a space from the class $\mathfrak{M}$. In the case $\tau = \aleph_0$ we shall call the property Q not $\aleph_0$ -factorizable, but σ -factorizable.

Lemma on σ . *If a property Q of a system of sets of a space X is (strongly, weakly) factorizable with respect to the class $\mathfrak{M}_c$ of spaces with a countable base, then the property Q is (strongly, weakly) σ -factorizable with respect to the class of all spaces with a countable base and, consequently, with respect to the class of all metric spaces (in the latter case without narrowing).*

Definition. We shall say that a system π of normal subsets A_α , $\alpha \in \mathfrak{A}$, of a space X : a) is **dim-closed**; b) **has type F_σ with respect to the dimension dim**, if respectively: a) for the closure $[A_n]$ of the sum A_n of all sets A_α with $\dim A_\alpha \leq n$, the relation $\dim[A_n] \leq n$, $n = 0, 1, 2, \dots$, holds; b) the sum A_n of all sets A_α with $\dim A_\alpha \leq n$ has type F_σ in X and $\dim A_n \leq n$.

It is clear that a locally finite system of closed sets of a space X is dim-closed, and a σ -locally finite system of sets of type F_σ will have type F_σ with respect to the dimension dim. It is also clear that

* $g\tilde{A}_\alpha$ is some normal subset of the space Z containing gA_α .

** Consequently, the set $h(Z)$ is closed in Y .

a system π decomposing into a countable sum of systems having type F_σ with respect to the dimension dim, will itself be of the same kind. Thus, if the property of a system of sets of having type F_σ with respect to the dimension dim turns out to be factorizable in some sense with respect to a class of spaces $\mathfrak{M}$, then this property will also be σ -factorizable (in the corresponding sense) with respect to the class $\mathfrak{M}$.

Lemma. A σ -strongly locally countable system π of sets of type F_σ in a space X has type F_σ with respect to the dimension dim.

Theorem 1 follows from the following assertion.

Proposition 1. a) The property of a system π of subsets of a space X of being dim-closed is strongly σ -factorizable without enlargement with respect to the class of metric spaces. b) The property of a system π of sets of spaces X of having type F_σ with respect to the dimension dim is strongly* σ -factorizable without enlargement with respect to the class of metric spaces.

It is possible to factor not only one mapping into a metric space, but any countable family of such mappings:

Theorem 2. a) If a system π of subsets A_α , $\alpha \in \mathfrak{A}$, of a space X decomposes into a countable sum of dim-closed subsystems π_j , $j = 1, 2, \dots$, then for any countable collection of mappings f_i , $i = 1, 2, \dots$, of the space X into metric spaces

R_i , there exists a metric space S and mappings $g : X \rightarrow S$ and $h_i : S \rightarrow R_i$ such that: 1) $f_i = h_i \cdot g$, $i = 1, 2, \dots$; 2) $w(B) \leq \max_i(w(h_i B))$ for every set $B \subseteq S$, in particular $w(S) \leq \max_i(w(R_i))$; 3) $\dim S \leq \dim X$, $\dim[gA_\alpha] \leq \dim A_\alpha$; 4) the product $h : S \rightarrow \prod R_i$ of the mappings h_i is perfect; in particular, if all the spaces R_i are complete, then S will also be complete.

- b) If in part a) the systems π_j , $j = 1, 2, \dots$, are taken not to be dim-closed, but to have type F_σ with respect to the dimension $\dim$, then assertions 1), 2) and 4) remain valid, and instead of assertion 3) the following assertion is true: 3') $\dim S \leq \dim X$, and for every $\alpha \in \mathfrak{A}$ the set gA_α is contained in a set B_α of type F_σ in S with $\dim B_\alpha \leq \dim A_\alpha$.

From Theorem 2 and the theorem of § 1 of [7] it follows that

Theorem 3. 1) For any σ -locally finite (σ -locally countable) system π of closed (respectively, of type F_σ) subsets A_α , $\alpha \in \mathfrak{A}$, of a metrizable space X ; 2) for any countable system of homeomorphisms h_i , $i = 1, 2, \dots$, of the space X onto itself and 3) for any countable system of completions $\tilde{X}_j$, $j = 1, 2, \dots$, of the space X with respect to some metrics on X , there exists a metric on X such that for the completion $\tilde{X}$ of the space X with respect to this metric the following holds: 1) $\dim \tilde{X} = \dim X$, $\dim A_\alpha = \dim[A_\alpha]_{\tilde{X}}$, $\alpha \in \mathfrak{A}$ (respectively, $\dim \tilde{X} = \dim X$, and for each $\alpha \in \mathfrak{A}$ the set A_α is contained in a set B_α of type F_σ in $\tilde{X}$ with $\dim B_\alpha \leq \dim A_\alpha$); 2) the homeomorphisms h_i extend to homeomorphisms $\tilde{h}_i$ of the completion $\tilde{X}$ onto itself, $i = 1, 2, \dots$; 3) for every $j = 1, 2, \dots$ there exists a mapping of $\tilde{X}$ into $\tilde{X}_j$ which is the identity on X .

This theorem is, in a certain sense, a generalization of Theorem II.10 of [8].

Theorem 3 permits the following strengthening of the theorem on approximation of an n -dimensional space, complete in the sense of Dieudonné, by a spectrum of n -dimensional metric spaces.

Theorem 4. Let the space X be complete in the sense of Dieudonné, and let a system π of sets A_α , $\alpha \in \mathfrak{A}$, of type F_σ in X be σ -strongly locally countable (if X is paracompact, then the system π may be merely σ -locally countable). Then there exists a spectrum $S = \{R_\lambda, \mathfrak{F}_\lambda^\mu\}$; $\lambda \in \Lambda$, of metric spaces R_λ with projections “onto,” such that: 1) the space X is the limit of the spectrum S and $\dim R_\lambda \leq \dim X$, $\lambda \in \Lambda$; 2) for every $\alpha \in \mathfrak{A}$ the set A_α is the limit of the spectrum $S_\alpha = \{\mathfrak{F}_\lambda A_\alpha, \mathfrak{F}_\lambda^\mu\}$, $\lambda \in \Lambda$ ($\mathfrak{F}_\lambda$

denotes the projection of the limit X of the spectrum S onto the element R_λ of this spectrum) and $\dim \mathfrak{F}_\lambda A_\alpha \leq \dim A_\alpha$, $\lambda \in \Lambda$.

Remark. In fact, not only $\dim \mathfrak{F}_\lambda A_\alpha \leq \dim A_\alpha$, $\lambda \in \Lambda$, but in each space R_λ , for every α there exists a set B_α^λ of type F_σ in R_λ such that $B_\alpha^\lambda \supset \pi_\lambda A_\alpha$ and $\dim B_\alpha^\lambda \leq \dim A_\alpha$. If, moreover, the system π consists of closed sets and is σ -locally finite, then one may take $\dim[\pi_\lambda A_\alpha] \leq \dim A_\alpha$, $\lambda \in \Lambda$, $\alpha \in \mathfrak{A}$.

II. A zero-dimensional perfect mapping onto a metric space will be called

an **improvement**, and spaces possessing an improvement will be called **improvable**. The class of improvable spaces is in many respects similar to the class of metric spaces, as is shown, in particular, by the following assertion.

Reduction Lemma 2. *If the system π of subsets A_α , $\alpha \in \mathfrak{A}$, of the space X : a) is weakly factorizable, b) factorizable, c) strongly factorizable, or d) strongly* factorizable with respect to the class $\mathfrak{M}$ of metric spaces, then the system π is factorizable without restriction in the corresponding sense (a), b), c), or d)) with respect to the class $\mathfrak{M}_a$ of improvable spaces. In this case the sets $\tilde{g}A_\alpha$ in case a) and B_α in case d) (see the basic definitions) may be considered improvable spaces.*

From Reduction Lemma 2 it follows immediately that

Assertion 1. *Proposition 1 and Theorem 1 remain valid if in them the class of metric spaces is replaced by the class of improvable spaces.*

Since the product of a countable system of improvements is an improvement, it follows that

Assertion 2. *Theorem 3 remains valid if in it the spaces R_i , $i = 1, 2, \dots, S$, and the sets B_α are regarded as improvable.*

The analogue of Theorem 2 is also valid:

Assertion 3. *1) For any σ -locally finite (σ -locally countable) system π of closed (respectively, of type F_σ) subsets A_α , $\alpha \in \mathfrak{A}$, of an improvable space X , and 2) for any countable system of complete in the sense of Čech improvable extensions $\tilde{X}_j$, $j = 1, 2, \dots$, of an improvable space X , there exists a complete in the sense of Čech improvable extension $\tilde{X}$ of the space X such that: 1) $\dim \tilde{X} = \dim X$, $\dim[A_\alpha]_{\tilde{X}} = \dim A_\alpha$, $\alpha \in \mathfrak{A}$ (respectively, for each α the set A_α is contained in a set B_α of type F_σ in $\tilde{X}$ with $\dim B_\alpha \leq \dim A_\alpha$); 2) for any $j = 1, 2, \dots$ there exists a mapping of $\tilde{X}$ into $\tilde{X}_j$, identical on X .*

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