

# ON MONOTONE SOLUTIONS OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS OF THE $(n)$ -TH ORDER

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**Abstract**

**Full Text**

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**MATHEMATICS**

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# ON MONOTONE SOLUTIONS OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS OF THE $n$ -TH ORDER

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In the present paper we study the problem of the existence of a solution  $u(t)$  of the differential equation

$$u^{(n)} = f(t, u, u', \dots, u^{(n-1)}), \quad (1)$$

defined on the interval  $(0, +\infty)$  and satisfying the conditions

$$\lim_{t \rightarrow 0} u(t) = u_0, \quad (-1)^k u^{(k)}(t) \geq 0 \quad \text{for } t > 0 \quad (k = 0, 1, \dots, n-1). \quad (2)$$

Previously this problem had been investigated only for the case  $n = 2$  (<sup>1-5</sup>).

By a solution of problem (1)–(2) we mean a function  $u(t)$ , absolutely continuous together with its derivatives up to order  $n - 1$  inclusive on every segment contained in the interval  $(0, +\infty)$ , and satisfying equation (1) and conditions (2) on this interval.

Before proceeding to the formulation of the existence theorem, we give some definitions and one auxiliary proposition.

**Definition 1.**  $\omega(t, x) \in B_n(r; a)$  if  $\omega(t, x)$  is defined and nonnegative in the domain  $t \in (0, a]$ ,  $x \in [0, +\infty)$ , and there exist a number  $a_0 \in (0, a)$  and a function  $b(t)$ , continuous on the half-interval  $(0, a]$ , with  $t^{n-2}b(t) \in L(0, a)$ , such that, for any  $\alpha \in (0, a_0]$ , for every function  $u(t)$  absolutely continuous together with its derivatives up to order  $n - 1$  inclusive on the interval  $[\alpha, a]$  and satisfying on this interval the inequalities

$$(-1)^k u^{(k)}(t) \geq 0 \quad (k = 0, 1, \dots, n), \quad u(t) \leq r, \quad |u^{(n)}(t)| \leq \omega(t, |u^{(n-1)}(t)|),$$

we have  $|u^{(n-1)}(t)| \leq b(t)$  for  $t \in [\alpha, a]$ .

**Definition 2.**  $\psi(t, x_1, \dots, x_m) \in K_t(\alpha, \beta)$  if  $m \geq 1$ ,  $\psi(t, x_1, \dots, x_m)$  is measurable in  $t$  on the interval  $(\alpha, \beta)$  for every  $(x_1, \dots, x_m) \in R_m$ , continuous in

$(x_1, \dots, x_m)$  in the space  $R_m$  for almost all  $t \in (\alpha, \beta)$ , and for every  $r > 0$  there exists a function  $\psi(t; r) \in L(\alpha, \beta)$  such that

$$|\psi(t, x_1, \dots, x_m)| \leq \psi(t; r)$$

for  $t \in (\alpha, \beta)$ ,  $|x_k| \leq r$  ( $k = 1, 2, \dots, m$ ) (local Carathéodory conditions).

In what follows it is everywhere assumed that  $n \geq 2$  and  $f(t, x_1, \dots, x_n) \in K_t(\alpha, \beta)$  for arbitrary  $\alpha, \beta \in (0, +\infty)$ ; here the case is not excluded when the function  $f(t, x_1, \dots, x_n)$ , having a singularity at  $t = 0$ , does not belong to the set  $K_t(0, \beta)$ .

**Lemma.** If  $n \geq 3$ , the function  $u(t)$  is absolutely continuous together with its derivatives up to order  $n - 1$  inclusive on the interval  $[\alpha, \beta]$ ,  $u^{(k)}(\beta) = 0$  ( $k = 1, \dots, n - 2$ ), and

$$(-1)^k u^{(k)}(t) \geq 0$$

for  $t \in [\alpha, \beta]$  ( $k = 0, 1, \dots, n$ ), then for  $t \in [\alpha, \beta]$  the inequalities

$$|u^{(k-1)}(t)| \leq \frac{[(n-1)!]^{(n-k)/(n-1)}}{(n-k)!} |u(t)|^{(n-k)/(n-1)} |u^{(n-1)}(t)|^{(k-1)/(n-1)}$$

$$(k = 1, 2, \dots, n).$$

**Theorem 1.** If

$$f(t, 0, \dots, 0) \equiv 0, \quad (-1)^n f(t, x_1, \dots, x_n) \geq 0$$

for  $t > 0$ ,  $(-1)^{k-1} x_k \geq 0$  ( $k = 1, 2, \dots, n$ ),  
 $a > 0$ ,  $r > 0$ , and in the domain

$$t \in (0, a], \quad 0 \leq x_1 \leq r, \quad 0 \leq (-1)^{k-1} x_k \leq$$

$$\leq \frac{[(n-1)! x_1]^{(n-k)/(n-1)}}{(n-k)!} |x_n|^{(k-1)/(n-1)}$$

$$(k = 2, \dots, n-1), \quad (-1)^{n-1} x_n \geq 0 \quad (4)$$

the inequality

$$|f(t, x_1, \dots, x_n)| \leq \omega(t, |x_n|) \quad (5)$$

is satisfied, where  $\omega(t, x) \in B_n(r; a)$ , then for every  $u_0 \in [0, r]$  the problem (1)–(2) has at least one solution.

We indicate the scheme of the proof of Theorem 1. Let the number  $a_0$  and the function  $b(t)$  be chosen according to Definition 1. Put

$$b_k(t) = (n-1)! r (a - a_0)^{-k} + \int_t^a \tau^{n-2-k} b(\tau) d\tau \quad (k = 0, \dots, n-2),$$

$$b_{n-1}(t) = b(t); \quad (6)$$

$$\rho(t) = \max_{t \leq \tau \leq a} \sum_{k=0}^{n-1} b_k(\tau); \quad (7)$$

$$\chi(t; \tau) = \begin{cases} 1, & \text{for } 0 \leq t \leq \tau, \\ 2 - t/\tau, & \text{for } \tau < t < 2\tau, \\ 0, & \text{for } t \geq 2\tau; \end{cases} \quad \sigma_k(t) = \begin{cases} t, & \text{for } (-1)^{k-1}t > 0, \\ 0, & \text{for } (-1)^{k-1}t \leq 0; \end{cases} \quad (8)$$

$$g_m(t, x_1, \dots, x_n) = \chi\left(\sum_{i=1}^n |x_i|; \rho\left(\frac{a_0}{m}\right)\right) f(t, \sigma_1(x_1), \dots, \sigma_n(x_n)). \quad (9)$$

Since  $t^{n-2}b(t) \in L(0, a)$ , it is clear from (6) that

$$b_1(t) \in L(0, a). \quad (10)$$

Taking into account conditions (3), with the aid of Schauder's theorem it is easy to prove that, for every natural number  $m$ , the differential equation

$$u^{(n)} = g_m(t, u, u', \dots, u^{(n-1)}) \quad (11)$$

has a solution  $u_m(t)$  satisfying the boundary conditions

$$u_m(a_0/m) = u_0, \quad u_m^{(k)}(a+m) = 0 \quad (k = 0, 1, \dots, n-2) \quad (12)$$

and the inequalities

$$(-1)^k u_m^{(k)}(t) \geq 0 \quad \text{for } t \in \left[\frac{a_0}{m}, a+m\right] \quad (k = 0, 1, \dots, n). \quad (13)$$

Since  $u_0 \in [0, r]$ , according to (12) and (13), from the lemma cited above it is clear that, for  $t \in [a_0/m, a+m]$ , the point  $(t, u(t), \dots, u^{(n-1)}(t))$  belongs to the domain (4); therefore, by virtue of (5), (8), and (9), from (11) we have

$$|u_m^{(n)}(t)| \leq \omega(t, |u_m^{(n-1)}(t)|) \quad \text{for } t \in [a_0/m, a].$$

Hence, according to Definition 1 and conditions (6), (7), (10), (12), and (13), we easily find that

$$|u_m^{(k)}(t)| \leq b_k(t) \quad \text{for } t \in \left[\frac{a_0}{m}, a\right] \quad (k = 0, 1, \dots, n-1),$$

$$\sum_{k=0}^{n-1} |u_m^{(k)}(t)| \leq \rho(a_0/m) \quad \text{for } t \in \left[\frac{a_0}{m}, a+m\right]; \quad (14)$$

$$|u_m(t) - u_0| \leq \int_0^t b_1(\tau) d\tau \quad \text{for } t \in \left[\frac{a_0}{m}, a+m\right]. \quad (15)$$

By virtue of (8), (9), and (14), it follows from (11) that  $u_m(t)$  is a solution of equation (1) on the interval  $[a_0/m, a+m]$ .

Relying on inequalities (13) and (14), one can show that the sequence  $\{u_m(t)\}$  contains a subsequence  $\{u_{m_i}(t)\}$  such that  $\{u_{m_i}^{(k)}(t)\}$  ( $k = 0, 1, \dots, n-1$ ) converges uniformly on every segment contained in  $(0, +\infty)$ .

It is not hard to prove that  $u(t) = \lim_{i \rightarrow \infty} u_{m_i}(t)$  is a solution of equation (1), defined on the interval  $(0, +\infty)$ , and, on the other hand, it follows directly from (13) and (15) that  $u(t)$  satisfies conditions (2).

**Corollary 1.** *Suppose that conditions (3) are satisfied, and that in the domain (4) inequality (5) holds, where  $\omega(t, x) \geq 0$  for  $t \in (0, a)$ ,  $x \geq 0$ ,  $\omega(t, x) \in K_t(a, a)$  for any  $a \in (0, a)$ , and there exists a positive number  $\rho_0$  such that the upper solution  $\rho(t)$  of the problem*

$$d\rho/dt = -\omega(t, \rho), \quad \rho(a) = \rho_0$$

*is defined on the interval  $(0, a]$  and*

$$(n-2)!r < \int_0^a t^{n-2} \rho(t) dt < +\infty.$$

*Then problem (1)–(2) is solvable for any  $u_0 \in [0, r]$ .*

The condition of Corollary 1 is satisfied, for example, by the function  $\omega(t, x) = \psi(t)\omega(x)$ , if  $\psi(t) > 0$  for  $t \in (0, a)$ ,  $\omega(x)$  is positive and continuous on the interval  $(0, +\infty)$ , and either

$$\int_0^a \psi(t) dt < \int_0^{+\infty} \frac{dt}{\omega(t)} < +\infty, \quad \int_0^a t^{n-2} \Omega^{-1} \left[ \int_0^t \psi(\tau) d\tau \right] dt > (n-2)!r,$$

where  $\Omega^{-1}(t)$  is the function inverse to

$$\Omega(t) = \int_t^{+\infty} \frac{d\tau}{\omega(\tau)},$$

or

$$\Omega_\delta(t) = \int_\delta^t \frac{d\tau}{\omega(\tau)} \rightarrow +\infty \quad \text{as } t \rightarrow +\infty, \quad t^{n-2} \Omega_\delta^{-1} \left[ \int_t^a \psi(\tau) d\tau \right] \in L(0, a),$$

where

$$\delta > (n-1)! a^{1-n} r,$$

and  $\Omega_\delta^{-1}(t)$  is the function inverse to  $\Omega_\delta(t)$ .

From Corollary 1 one easily obtains the following simple

**Corollary 2.** Suppose that conditions (2) are satisfied,  $a > 0$ ,  $r > 0$ , and in the domain  $t \in (0, a)$ ,  $(-1)^{k-1} x_k \geq 0$  ( $k = 1, 2, \dots, n$ ), we have

$$|f(t, x_1, \dots, x_n)| \leq A t^{(n-1)\lambda-n} |\ln t|^{\lambda-1-\varepsilon} \sum_{k=2}^n (1 + |x_n|)^{\mu_k} (1 + |x_k|)^{\frac{n-1}{k-1} \nu_k},$$

where  $A \geq 0$ ,  $\nu_k \geq 0$ ,  $\mu_k + \nu_k \leq \lambda$  ( $k = 2, \dots, n$ ),  $\varepsilon > 0$  for  $\lambda \leq 1$  and  $\varepsilon = 0$  for  $\lambda > 1$ . Then problem (1)–(2) is solvable for any  $u_0 \in [0, r]$ .

It should be noted that the above-stated conditions of the form (5) are, in a certain sense, close to necessary conditions. As examples, let us consider the equations

$$u^{(n)} = (-1)^n t^{(n-1)\lambda-n} (1 + |\ln t|)^\nu (1 + |u^{(n-1)}|)^\lambda u; \quad (16)$$

$$u^{(n)} = (-1)^n t^{(n-1)\mu-n} (1 + |\ln t|)^\eta (u + |u^{(n-1)}|)^\mu, \quad (17)$$

where  $\lambda \leq 1$ ,  $\mu > 1$ . It follows directly from Corollary 2 of Theorem 1 that, if  $\nu < \lambda - 1$  and  $\eta \leq \mu - 1$ , then, for any  $u_0 \geq 0$ , equations (16) and (17)

have solutions satisfying conditions (2); on the other hand, one can show that if  $\nu \geq \lambda - 1$ , then problem (16)–(2) has no solution for any  $u_0 > 0$ , while if  $\eta > \mu - 1$ , then problem (17)–(2) has no solution for sufficiently large positive  $u_0$ .

**Theorem 2.** *If  $f(t, x_1, \dots, x_n) \in K_t(0, a)$  for any  $a \in (0, +\infty)$  and conditions (3) are satisfied, then there exists a positive number  $r$  such that problem (1)–(2) is solvable for any  $u_0 \in [0, r]$ .*

In conclusion we give two theorems on the behavior of solutions of problem (1)–(2) as  $t \rightarrow +\infty$ .

**Theorem 3.** Suppose that conditions (3) hold,  $a > 0$ ,  $r > 0$ ,  $\varepsilon > 0$ , and in the domain  $t \in [a, +\infty)$ ,  $0 \leq x_1 \leq r$ ,  $0 \leq (-1)^{k-1}x_k \leq \varepsilon$  ( $k = 2, \dots, n$ ) we have

$$|f(t, x_1, \dots, x_n)| \geq \sigma_1(t, x_1) + \sigma_2(t, x_1)|x_k|,$$

where  $\sigma_k(t, x_1)$  ( $k = 1, 2$ ) are functions nondecreasing in  $x_1$ ,  $\sigma_k(t, x_1) \in K_t(0, \alpha)$  for every  $\alpha \in (a, +\infty)$  ( $k = 1, 2$ ), and for every  $x_1 \in (0, r]$

$$\lim_{s \rightarrow +\infty} \int_a^s t^{n-2} \left\{ \int_t^s \sigma_1(\tau, x_1) \exp \left[ \int_t^\tau \sigma_2(\xi, x_1) d\xi \right] d\tau \right\} dt = +\infty.$$

Then, whatever  $u_0 \in [0, r]$ , every solution  $u(t)$  of problem (1)–(2) satisfies the conditions

$$\lim_{t \rightarrow +\infty} t^k u^{(k)}(t) = 0 \quad (k = 0, 1, \dots, n-1).$$

**Theorem 4.** Suppose that conditions (3) hold,  $r > 0$ , and, for any  $x > 0$ ,  $a > 0$ , the function  $f(t, x, 0, \dots, 0)$  is different from zero on a set of positive measure of the interval  $[a, +\infty)$ , and for sufficiently large  $t$ , in the domain (4), we have

$$|f(t, x_1, \dots, x_n)| \geq \delta t^{(k-1)\lambda-n} |x_k|^\lambda,$$

where  $1 \leq k \leq n$ ,  $\delta > 0$ ,  $0 < \lambda < 1$ . Then, whatever  $u_0 \in [0, r]$ , for every solution  $u(t)$  of problem (1)–(2) there exists a number  $t_0$  such that

$$u(t) \equiv 0 \quad \text{for } t \geq t_0.$$

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