

SCATTERING OF A PLANE WAVE BY A SINGULAR POTENTIAL

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Abstract

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MATHEMATICS

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SCATTERING OF A PLANE WAVE BY A SINGULAR POTENTIAL

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Numerous works have been devoted to quantum scattering theory in the stationary formulation; closest to us in subject matter are the works ^(1,2). In all the works the potential $V(x)$ was assumed to be an integrable function, since otherwise the Lippmann–Schwinger equation, which served as the principal tool of investigation in ^(1,2), has an essential singularity (some special cases of singularities were studied by L. G. Mikhailov ⁽³⁾). However, from the physical point of view the assumption of integrability of the potential $V(x)$ is too restrictive: there exist problems in which the potential $V(x)$ becomes $+\infty$ on a set of positive measure. We propose such an integral equation whose kernel is a bounded integrable function independently of the dimension of the space N and of the singularity of the potential $V(x)$.

Let R_N be N -dimensional Euclidean space ($N \geq 3$); Δ the Laplace operator; $V(x)$ a scalar function (potential), $0 \leq V(x) \leq \infty$; $\Omega = \{x; V(x) = +\infty\}$. $\tilde{H}$ is the operator defined on functions finite in $R_N \setminus \Omega$ by the formula $\tilde{H}u = -\Delta u + V(x)u$.

Put $V_M(x) = \min\{V(x), M\}$ ($M \geq 0$), and in what follows we shall mark by the index M all quantities referring to the potential $V_M(x)$.

Below we assume everywhere that the potential $V(x)$ satisfies the following two conditions: 1) every function $V_M(x)$ is nonnegative and satisfies the Hölder condition (locally); 2) there exist constants $a > 0$, $C < \infty$, $R < \infty$, independent of M , such that for all x lying outside the ball of radius R , the estimate $V(x) \leq C|X|^{-N-a}$ holds.

Consider the Cauchy problem for the heat equation (in the whole space):

$$u(x, 0) = u_0(x), \quad \partial u / \partial t = -\tilde{H}_M u, \quad x \in R_N, \quad t > 0, \quad u \in L^\infty, \quad (1)$$

and let $G_M(x, y, t)$ be the Green function of problem (1). Represent it in the form:

$$G_M(x, y, t) = G_0(x, y, t) - g_M(x, y, t);$$

$$G_0(x, y, t) = (4\pi t)^{-N/2} \exp\{-(x - y)^2/4t\}.$$

Lemma 1. 1. For any $t > 0$ there exists the limit

$$\lim_{M \rightarrow \infty} g_M(x, y, t) = g(x, y, t).$$

2. In the uniform operator topology of the space $[L^p \rightarrow L^q; 1 \leq p \leq \infty, 1 \leq q < \infty]^*$ the operators $\hat{g}_M(t)^{**}$ converge to the operator $\hat{g}(t)$.

3. The operator $\hat{g}(t)$ is completely continuous as an operator from $[L^p \rightarrow L^q, 1 < p \leq \infty, 1 \leq q < \infty]$.

* $[A_0 \rightarrow B, \pi]$ is the space of all bounded linear operators from a Banach space A into a Banach space B , where A and B satisfy condition π .

** $\hat{g}(t)$ is the integral operator with kernel $g(x, y, t)$.

Corollary. The operators $\hat{G}_M(t)$ converge to the operator $\hat{G}(t)$; the operator function $\hat{G}(t)$ is a semigroup in $L^p, 1 < p < \infty$.

It can be verified that in $L^p(R_N \setminus \Omega)$ the operator function $G(t)$ is a semigroup of class C_0 .

Convergence of the operators $\hat{G}_M(t)$ to $\hat{G}(t)$ in the metric $[L^\infty \rightarrow L^\infty]$ need not hold; however, $\hat{G}(t)$ is a semigroup in L^∞ . Let $\hat{A}$ be the infinitesimal operator of the semigroup $\hat{G}(t)$, $Hu = -\hat{A}u$. The operator H is an extension of the operator $\hat{H}$.

The problem that we have to solve is formulated as follows: to find that solution of the equation

$$Hu = \lambda u, \quad x \in R_N, \quad u \in L^\infty, \quad \lambda > 0, \quad (2)$$

which can be represented in the form

$$u(x, k) = e^{ikx} + \varphi(x, k), \quad (3)$$

where $k^2 = \lambda$, and the function $\varphi(x, k)$, as $|x| \rightarrow \infty$, satisfies the estimates

$$\varphi(x, k) = O(|x|^{(1-N)/2}) \left(\frac{\partial}{\partial |x|} - i|k| \right) \varphi(x, k) o(|x|^{(1-N)/2}), \quad |x| \rightarrow \infty. \quad (4)$$

Consider the equation

$$e^{-\lambda t}u = \hat{G}(t)u. \quad (5)$$

Substituting (3) into (5), we obtain that the function $\varphi(x, k)$ satisfies the equation

$$(e^{-\lambda t} - \hat{G}_0)\varphi = \hat{g}(t)\varphi.$$

Let $K^+(\lambda)$ be the integral operator with kernel

$$K^+(\lambda, r_{xy}) = \left(\frac{1}{2\pi}\right)^{N/2} \lim_{\varepsilon \rightarrow +0} \int_0^\infty \frac{e^{-\rho^2 t}}{e^{-(\lambda+i\varepsilon)t} - e^{-\rho^2 t}} \frac{J_{N/2\alpha-1}(r_{xy}\rho)}{(r_{xy})^{N/2\alpha-1}} \rho^{N/2} d\rho$$

$$(r_{xy} = |x - y|).$$

Lemma 2. If $\varphi(x) \in L^p \cap L^\infty$, where $p > 2N/(N-1)$, then

$$(e^{-\lambda t} - G_0)e^{\lambda t}(E + K^+(\lambda))\varphi = e^{\lambda t}(E + \hat{K}^+(\lambda))(e^{-\lambda t} - G_0)\varphi = \varphi.$$

Put, by definition,

$$\hat{T}^+(\lambda) = -e^{\lambda t}(E + \hat{K}^+(\lambda))\hat{g}, \quad \hat{T}_M^+(\lambda) = -e^{\lambda t}(E + \hat{K}^+(\lambda))\hat{g}_M$$

and consider the equations

$$\psi(x, k, \lambda) = (\hat{T}^+(\lambda)(e^{iky} + \psi))(x, k, \lambda), \quad 0 < \lambda < \infty, \quad k \in R_N; \quad (6)$$

$$\psi_M(x, k, \lambda) = (\hat{T}_M^+(\lambda)(e^{iky} + \psi_M))(x, k, \lambda), \quad 0 < \lambda < \infty, \quad k \in R_N. \quad (7)$$

The number $t > 0$ enters these equations as a parameter.

Let Ω_1 be the largest open connected set contained in the set $R_N \setminus \Omega$ and containing points of the sphere $|x| = 2R$; $\Omega_2 = R_N \setminus (\Omega \cup \Omega_1)$.

The main result of our work is formulated in Theorems 1-3.

Theorem 1. There exists a countable set of points $\{\lambda_i\}$, independent of $t > 0$, having the property that for each λ_i the equation

$$H\varphi = \lambda_i\varphi$$

has m_i , $1 \leq m_i < \infty$, nontrivial solutions; all these solutions belong to L^∞ , vanish outside the set Ω_2 (whence it follows that if $\text{mes } \Omega_2 = 0$, then the set $\{\lambda_i\}$ is empty) and satisfy the equality $T^+(\lambda_i)\varphi = \varphi$, while for any $\lambda \in (0, \infty) \setminus \{\lambda_i\}$ the operator $(E - T^+(\lambda))^{-1} \in [L^q \rightarrow L^q, 2N/(N-1) < q \leq \infty]$.

Theorem 2. The set of points $\{\lambda_i\}$ discussed in Theorem 1 has the following properties: it is located on the positive ray from the point 0, and this distance is bounded below by a quantity depending only on the measure of the domain Ω_2 ; moreover, the number of points of the set $\{\lambda_i\}$ lying in the interval $(0, \lambda)$ satisfies the asymptotic estimate

$$N(\lambda) = \sum_{\lambda_i \leq \lambda} 1 \leq \frac{\text{mes } \Omega_2}{(4\pi)^{N/2} \Gamma(N/2 + 1)} \lambda^{N/2} + o(\lambda^{N/2}).$$

Theorem 3. If $\lambda \notin \{\lambda_i\}$, then there exists a solution of problem (2)–(4), namely

$$\varphi(x, k) = \psi(x, k, k^2),$$

where ψ is a solution of equation (6). For sufficiently small t , this solution exists, is unique, and is the limit as $M \rightarrow \infty$, in the metric L^q , $2N/(N-1) < q < \infty$, of solutions of equation (7).

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