

ON THE FUNDAMENTAL THEOREM OF HOMOLOGICAL DIMENSION THEORY

P. Aleksandrov

1968

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Abstract

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UDC 513.83

MATHEMATICS

Academician P. ALEKSANDROV

ON THE FUNDAMENTAL THEOREM OF HOMOLOGICAL DIMENSION THEORY

1. In this note the fundamental theorem of homological dimension theory—in a form directly adjoining the one originally given by me in 1930 (see ⁽¹⁾)—is proved for bcompacta (and consequently also for arbitrary normal spaces)*. The novelty consists in the fact that, with maximum generality, the formulation proposed here is extremely simple and transparent: I use only the concepts of a cycle and of homology on a finite simplicial complex and dispense with any inverse limits whatever. The proof rests only on the elementary case of Hopf's theorem on mappings of an n -dimensional polyhedron onto an n -dimensional complex (see, for example, ⁽²⁾, p. 70).
2. By a covering in this paper is always meant a finite open covering. The nerve of a covering ω is denoted by N_ω . If $\Phi \subset X$, then $N_{\omega\Phi}$ denotes the subcomplex of the nerve N_ω consisting of all simplices all vertices of each of which correspond to elements of the covering ω having common points with the set Φ . By the dimension of a space X we shall always mean the dimension $\dim X$, defined by means of coverings.

The following is proved.

Fundamental theorem (Theorem 1). *If a bcompactum X has (finite) dimension $\dim X = n > 0$, then there exist a closed set $\Phi \subset X$ and a covering ω of the space X such that for every covering ω' inscribed in ω the following assertions may be made:*

- (a) *On the nerve $N_{\omega'}$ there is an n -dimensional relative cycle $z_{\omega'}^n \bmod N_{\omega'\Phi}$ with respect to some (depending on ω') modulus $m = m_{\omega'}$, whose projection $\delta_{\omega'}^{\omega'} z_{\omega'}^n$ into the nerve N_ω is not homologous to zero on the complex $N_\omega \bmod N_{\omega\Phi}$.*
- (b) *On the subcomplex $N_{\omega'\Phi}$ of the nerve $N_{\omega'}$ there is an $(n-1)$ -dimensional cycle $z_{\omega'}^{n-1}$ modulo $m_{\omega'}$, homologous to zero on $N_{\omega'}$, and possessing the property that its projection $\delta_{\omega'}^{\omega'} z_{\omega'}^{n-1}$ is a cycle (of the complex $N_{\omega\Phi}$) not homologous to zero on $N_{\omega\Phi}$.***

Finally:

- (c) For $r > n$, for every covering ω of the space X there is a covering ω' inscribed in ω (of the space X) such that, for any closed $\Phi \subset X$, every r -dimensional relative cycle $z_{\omega'}^r$ on $N_{\omega'} \bmod N_{\omega'\Phi}$ and every $(r-1)$ -dimensional cycle $z_{\omega'}^{r-1}$ on $N_{\omega'\Phi}$, homologous to zero on $N_{\omega'}$, are equal to zero.

Assertion (c) is obvious: it suffices to take as ω' any covering inscribed in ω of multiplicity $n+1$.

3. We begin the proof of the main part of the theorem with some preliminary remarks.

By $\bar{Q}$ we shall denote once and for all an n -dimensional closed complex with center of gravity c and interior Q , lying in n -dimensional Euclidean space R^n , in turn embedded in some R^{2n+1} . Under

* The passage from normal spaces X to bicomplexa is effected automatically by means of the maximal bicomplex extension βX .

** From what has been said it evidently follows that $\delta_{\omega'}^{\omega'} z_{\omega'}^{n-1} \sim 0$ on $N_{\omega'}$ and, on the other hand, $z_{\omega'}^{n-1}$ is not ~ 0 on $N_{\omega'\Phi}$ (just as z_{ω}^n is not ~ 0 on $N_{\omega} \bmod N_{\omega\Phi}$).

By a “smaller” simplex ($\bar{Q}_0, \bar{Q}_1$, etc.) we shall always mean a simplex with the same center of gravity as, and homothetic to, the basic simplex $\bar{Q}$ (hence lying strictly inside Q).

Let φ be a continuous mapping of a once-and-for-all given n -dimensional bicomplexum X into the simplex $\bar{Q}$. Let $\omega = \{O_1, \dots, O_s\}$ be a covering of the bicomplexum X of multiplicity $n+1$. If $\varepsilon > 0$ is given, then by N_{ω} we shall denote the nerve of the covering ω , realized in the following way as a triangulation lying in $R^{2n+1} \supset R^n \supset \bar{Q}$. For each $i = 1, 2, \dots, s$ choose a point $p_i \in O_i$, and, moreover, if a closed set $\Phi \subset X$ has been distinguished, then for all $O_i \in \omega$ with $O_i \cap \Phi \neq \Lambda$ choose $p_i \in O_i \cap \Phi$. We choose the vertices $e_1, \dots, e_s$ of the nerve N_{ω} in R^{2n+1} in general position so that

$$\rho(e_i, \varphi p_i) < \varepsilon$$

(for $i = 1, 2, \dots, s$). A realization of the nerve N_{ω} satisfying these conditions will be called canonical (relative to the mapping φ and the number ε).

Assigning to each vertex e_i the point φp_i , we obtain a simplicial mapping g_{ω} of the nerve N_{ω} into $\bar{Q}$, which we shall call the canonical ε -shift (of the complex N_{ω}).

If the covering ω is normal (i.e. consists of open F_{σ} -sets), then by μ_{ω} we denote the barycentric mapping of the bicomplexum X into the body $\tilde{N}_{\omega}$ of the nerve N_{ω} .

4. Lemmas.

Lemma 1. *Let a mapping $\varphi \rightarrow \bar{Q}$ and a number $\varepsilon > 0$ be given. Then there exist an $\varepsilon' > 0$ and a covering $\omega = \{O_1, \dots, O_s\}$ of the bicomactum X such that, for any canonical realization of N_ω (relative to φ and ε') and the corresponding canonical ε' -shift $g_\omega : N_\omega \rightarrow \bar{Q}$, for every normal covering ω' inscribed in ω , the mapping $f = g_\omega \delta_\omega^{\omega'} \mu_{\omega'} : X \rightarrow \bar{Q}$ satisfies the inequality*

$$\rho(\varphi x, fx) < \varepsilon \quad \text{for all } x \in X.$$

Indeed, take $\varepsilon' < \varepsilon/3$ and open sets $V_1, \dots, V_\nu$ in the space R^n , covering the simplex $\bar{Q}$ and having diameters $< \varepsilon'$. Put $U_j = \varphi^{-1}V_j$ for $j = 1, 2, \dots, \nu$, and take a covering $\omega = \{O_1, \dots, O_s\}$ of the bicomactum X , of multiplicity $n+1$, inscribed in the covering $\{U_1, \dots, U_\nu\}$. Inscribe in ω any normal covering $\omega' = \{O'_1, \dots, O'_{s'}\}$ of multiplicity $n+1$.

Let the nerve N_ω be canonically realized at the vertices $e_1, \dots, e_s$, and the nerve $N_{\omega'}$ at the vertices $e'_1, \dots, e'_{s'}$ (both realizations are relative to the mapping φ and the number ε').

Take an arbitrary point $x \in X$; suppose it is contained in

$$O'_{k_0}, \dots, O'_{k_r}$$

and only in these elements of the covering ω' . Put

$$e_{i_\lambda} = \delta_\omega^{\omega'} e'_{k_\lambda}, \quad \lambda = 0, \dots, r.$$

Then

$$\mu_{\omega'} x \in |e'_{k_0} \dots e'_{k_r}| \in N_{\omega'}, \quad \delta_\omega^{\omega'} |e'_{k_0} \dots e'_{k_r}| = |e_{i_0} \dots e_{i_r}|,$$

so that

$$\delta_\omega^{\omega'} \mu_{\omega'} x \in |e_{i_0} \dots e_{i_r}|.$$

Further,

$$x \in O'_{k_\lambda} \subset O_{i_\lambda} \subset U_{j_\lambda}.$$

Hence

$$\rho(\varphi x, \varphi p_{i_\lambda}) < \varepsilon', \quad \rho(\varphi x, e_{i_\lambda}) < 2\varepsilon'$$

(all this for $\lambda = 0, \dots, r$). Since

$$\delta_\omega^{\omega'} \mu_{\omega'} x \in |e_{i_0} \dots e_{i_r}|,$$

we have

$$\rho(\varphi x, \delta_\omega^{\omega'} \mu_{\omega'} x) < 2\varepsilon',$$

and therefore

$$\rho(\varphi x, g_\omega \delta_\omega^{\omega'} \mu_{\omega'} x) < 3\varepsilon' < \varepsilon,$$

which proves Lemma 1.

Lemma 2. Let $\varphi : X \rightarrow \bar{Q}$ be an essential mapping. Then, for any smaller simplex $\bar{Q}_0 \subset \bar{Q}$, one can find a covering $\omega = \{O_1, \dots, O_s\}$ of the bicomcompactum X and a number $\varepsilon > 0$ such that, for an arbitrary normal covering $\omega' = \{O'_1, \dots, O'_{s'}\}$, of multiplicity $n + 1$, inscribed in ω , and for the canonical ε -shift $g_\omega : N_\omega \rightarrow \bar{Q}$, the simplicial mapping

$$\psi = g_\omega \delta_\omega^{\omega'} : \tilde{N}_{\omega'} \rightarrow \bar{Q}$$

essentially covers the simplex $\bar{Q}_0$ (i.e. the mapping of the polyhedron $\psi^{-1}\bar{Q}_0$ onto $\bar{Q}_0$ is essential).

The proof is based on the following slight modification of the main lemma for the theorem on essential mappings (see, for example, ⁽³⁾, Ch. 6, p. 217), whose proof we leave to the reader:

Lemma 2₀. With the notation adopted above, there is an $\varepsilon > 0$ such that every continuous mapping $f : X \rightarrow \bar{Q}$ differing from φ by less than ε essentially covers the simplex $\bar{Q}_0$.

But by Lemma 1 there exists a covering ω of the bicomcompactum X such that for every ω' inscribed in ω , for sufficiently small ε' , and for the corresponding canonical ε' -shift $g_\omega : N_\omega \rightarrow \bar{Q}$, we have the inequality

$$\rho(\varphi x, g_\omega \tilde{\omega}^{\omega'} \mu_{\omega'} x) < \varepsilon \quad \text{for all } x \in X.$$

Hence the mapping $f = g_\omega \tilde{\omega}^{\omega'} \mu_{\omega'} : X \rightarrow \bar{Q}$ essentially covers $\bar{Q}_0$. We show that then also the simplicial mapping $\psi = g_\omega \tilde{\omega}^{\omega'} : \tilde{N}_{\omega'} \rightarrow \bar{Q}$ essentially covers $\bar{Q}_0$. Let $X_0 = f^{-1}\bar{Q}_0 = \mu_{\omega'}^{-1}\psi^{-1}\bar{Q}_0$.

Put $Y = \psi^{-1}\bar{Q}_0 \subseteq \tilde{N}_{\omega'}$; then $X_0 = \mu_{\omega'}^{-1}Y$. Denoting by S_0 the boundary of the simplex Q_0 , put further $\Psi = \psi^{-1}S_0 \subset Y$. It is necessary to prove that the mapping $\psi : Y \rightarrow \bar{Q}_0$ is essential. Otherwise there is a mapping $\psi_1 : Y \rightarrow S_0$ for which $\psi_1 y = \psi y$ for every $y \in \Psi$.

We consider the barycentric mapping $\mu_{\omega'} : X_0 \rightarrow Y$ and define $\varphi_1 = \psi_1 \mu_{\omega'} : X_0 \rightarrow S_0$. For $x \in f^{-1}S_0$ we have $S_0 \ni fx = g_\omega \tilde{\omega}^{\omega'} \mu_{\omega'} x = \psi \mu_{\omega'} x$, i.e. $\mu_{\omega'} x \in \psi^{-1}S_0 = \Psi$, and, consequently, by the definition of ψ_1 we have $\psi_1 \mu_{\omega'} x = \psi \mu_{\omega'} x$, while by the definition of φ_1 we have $\varphi_1 x = \psi_1 \mu_{\omega'} x = \psi \mu_{\omega'} x = fx$. Thus we have a mapping $\varphi_1 : X_0 \rightarrow S_0$ coinciding with f on $f^{-1}S_0$, contrary to the essentiality of the mapping f . Lemma 2 is proved.

5. Proof of assertion (a) of Theorem 1. Keeping the notation of Lemmas 1 and 2, take smaller simplexes $\bar{Q}_1 \subset Q_0$ and $\bar{Q}_2 \subset Q_1$. Put $\Phi = \varphi^{-1}(\bar{Q} \setminus \bar{Q}_1) \subset X$. Suppose that ε in Lemma 2 is less than half the distance from each smaller simplex to its complement (in R^n) in the larger one. Then

$$f^{-1}S_0 \subseteq \Phi. \tag{1}$$

Otherwise we would have a point $x_0 \in X$ for which $fx_0 \in S_0$, but $\varphi x_0 \in Q_1$, and hence $\rho(\varphi x_0, fx_0) > \varepsilon$.

We derive from formula (1) the formula

$$\psi^{-1}S_0 \subseteq N_{\omega\Phi} \cap Y; \quad N_{\omega\Phi} \subseteq \psi^{-1}(Q \setminus Q_2). \quad (2)$$

We prove the first inclusion in (2). If* $y \in \psi^{-1}S_0$ and $x \in \mu_{\omega'}^{-1}y$, then $fx = \psi\mu_{\omega'}x \in S_0$, i.e. $x \in f^{-1}S_0 \subseteq \Phi$, and every $O'_k \in \omega'$ containing the point x intersects Φ .

Let x be contained in $O'_{k_0}, \dots, O'_{k_r}$, and only in these elements of the covering ω' . Then $y = \mu_{\omega'}x \in |e_{k_0} \dots e_{k_r}| \in N_{\omega\Phi}$. Thus, $\psi^{-1}S_0 \subseteq N_{\omega\Phi}$. Since, moreover, $\psi^{-1}S_0 \subseteq Y$, the first inclusion in formula (2) is proved.

We prove the second inclusion. Let $y \in |e_{k_0} \dots e_{k_r}| \in N_{\omega\Phi}$. We must reduce to a contradiction the inclusion $\psi y \in Q_2$. For this we take some $x \in \mu_{\omega'}^{-1}y$. The inclusion $\psi y \in Q_2$ means $fx \in Q_2$. Since $\rho(\varphi x, fx) < \varepsilon$, it follows that for every $O'_k \subset \omega'$ containing x , we have (recalling that $\text{diam } \varphi O'_k < \varepsilon$) the inclusions

$$\varphi O'_k \subset O(\varphi x, \varepsilon) \subseteq O(fx, 2\varepsilon) \subseteq Q_1,$$

and hence $O'_k \subseteq \varphi^{-1}Q_1$. On the other hand, the inclusion $y \in |e_{k_0} \dots e_{k_r}| \in N_{\omega\Phi}$ means that x is contained in $O'_{k_0}, \dots, O'_{k_r}$ and only in these elements of the covering ω' , and that each of these $O'_{k_0}, \dots, O'_{k_r}$ intersects $\Phi = \varphi^{-1}(\bar{Q} \setminus Q_1)$. The required contradiction has been obtained.

* We may always assume (replacing, if necessary, the covering ω' by an inscribed covering subordinate to it) that $\mu_{\omega'} : X \rightarrow N_{\omega'}$ is a mapping onto $N_{\omega'}$. In this case ω' is called irreducible if there is no covering inscribed in it whose nerve is a proper subcomplex of the nerve $N_{\omega'}$.

Remark. Let us prove the formula

$$g_{\omega}\tilde{N}_{\omega\Phi} \subseteq \bar{Q} \setminus Q_2. \quad (3_{\omega})$$

Indeed, the inclusion (3_{ω}) follows from the fact that all vertices of the complex $g_{\omega}N_{\omega\Phi}$ are points of the form φp_i , where $p_i \in O_i \cap \Phi$; that the simplices of this complex have diameter $< 2\varepsilon$, and that $\varphi\Phi \subseteq \bar{Q} \setminus Q_1$.

Let us sum up: we have an essential mapping $\psi = g_{\omega}\tilde{\mathfrak{d}}_{\omega'}^{\omega'}$ of the polyhedron Y (of dimension n) onto the n -dimensional complex $\bar{Q}_0$, under which $\psi^{-1}S_0$ is contained in the polyhedron $\tilde{N}_{\omega\Phi} \cap Y = \Pi_0 \subseteq \psi(\bar{Q}_0 \setminus Q_2)$. Denote now by $N_{\omega'}^{(1)}$ such a subdivision of the triangulation $\tilde{N}_{\omega'}$ that the polyhedron Y is the body $\tilde{K}$ of some complex $K \subseteq N_{\omega'}^{(1)}$, and the polyhedron Π_0 is the body of some complex $K_0 \subseteq K$.

We are in the conditions of the already mentioned theorem of Hopf ((²), p. 70), by virtue of which on the complex K there is a relative cycle $z_{\omega'_1}^n \bmod K_0$, with respect to some modulus $m = m_{\omega'}$, which covers under the mapping $\psi : Y \rightarrow \bar{Q}_0$ the point c with degree $\text{gr}_c \psi = \gamma \neq 0$. Under the canonical shift $\sigma : N_{\omega'}^{(1)} \rightarrow N_{\omega'}$ we have $\sigma K_0 \subseteq N_{\omega' \Phi}$, and the relative cycle $z_{\omega'_1}^n \bmod K_0$ passes into the relative cycle $\sigma z_{\omega'_1}^n = z_{\omega'}^n$ of the complex $N_{\omega'} \bmod N_{\omega' \Phi}$, while the boundary $z_{\omega'_1}^{n-1} = \Delta z_{\omega'_1}^n$ passes (remaining throughout in the polyhedron $\tilde{N}_{\omega' \Phi} \subseteq \bar{Q} \setminus Q_2$) into the cycle $z_{\omega'}^{n-1}$ of the complex $N_{\omega'}$. At the same time the cycle $\psi z_{\omega'_1}^{n-1}$ undergoes a deformation in $\bar{Q} \setminus Q_2$ which does not change the linking coefficient $\nu(c, \psi z_{\omega'_1}^{n-1}) = \text{gr}_c \psi z_{\omega'_1}^n$, so that $\text{gr}_c \psi z_{\omega'}^n = \text{gr}_c \psi z_{\omega'_1}^n = \gamma \neq 0$. Obviously, $\tilde{\mathfrak{d}}_{\omega'}^{\omega'} z_{\omega'}^n = z_{\omega'}^n$ is a relative cycle of the complex $N_{\omega'} \bmod N_{\omega' \Phi}$, with respect to the same modulus $m = m_{\omega'}$, and $\text{gr}_c g_{\omega} z_{\omega'}^n = \text{gr}_c g_{\omega} \tilde{\mathfrak{d}}_{\omega'}^{\omega'} z_{\omega'}^n = \text{gr}_c \psi z_{\omega'}^n = \gamma \neq 0$. Consequently, $z_{\omega'}^n$ is not ~ 0 on $N_{\omega'} \bmod N_{\omega' \Phi}$, as was required to prove.

6. Proof of assertion (6) of Theorem 1. We keep our notation. The projection $\mathfrak{d}_{\omega'}^{\omega'} z_{\omega'}^{n-1}$ is a cycle of the complex $N_{\omega' \Phi}$. Let us prove that it is not homologous to zero on $N_{\omega' \Phi}$. Otherwise, let $\mathfrak{d}_{\omega'}^{\omega'} z_{\omega'}^{n-1}$ be the boundary of a chain $x_{\omega'}^n$ of the complex $N_{\omega' \Phi}$. By formula (3 _{ω}), the chain $g_{\omega} x_{\omega'}^n$ lies in $\bar{Q} \setminus Q_2$, so that $\text{gr}_c g_{\omega} x_{\omega'}^n = 0$; therefore for the cycle $y_{\omega'}^n = z_{\omega'}^n - x_{\omega'}^n$ we have $\text{gr}_c g_{\omega} y_{\omega'}^n = \gamma \neq 0$, which is obviously impossible, since the cycle $g_{\omega} y_{\omega'}^n$, like every n -dimensional cycle on the n -dimensional complex $\bar{Q}_0$, is homologous in $\bar{Q}_0$ to zero. Theorem 1 is proved.

In the metric case Theorem 1 assumes, as is easy to establish, the following form:

Theorem 2. *For every compactum X of finite dimension $n \geq 1$ there exist a closed set $\Phi \subset X$ and an $\varepsilon > 0$ such that for every (arbitrarily small) $\varepsilon' > 0$ there is, first, an n -dimensional relative ε' -cycle on $X \bmod \Phi$, with respect to some (depending on ε') modulus $m_{\varepsilon'}$, not ε -homologous to zero on $X \bmod \Phi$, and, secondly, on Φ there is an $(n-1)$ -dimensional ε' -cycle $z_{\varepsilon'}^{n-1}$ modulo $m_{\varepsilon'}$, ε' -homologous to zero on X , but not even ε -homologous to zero on Φ .*

Here n is the greatest natural number for which at least one of these assertions holds.

The formulation of Theorem 2 (entirely in the spirit of Brouwer's classical works) is in an obvious way equivalent to my main theorem in its original formulation (paper (¹), p. 195).

Mechanical-Mathematical Faculty
of Moscow State University
named after M. V. Lomonosov

Received
15 II 1968

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Note: Figure translations are in progress. See original paper for figures.

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