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Abstract

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MATHEMATICS

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ON THE IDENTITY PROBLEM FOR FINITELY PRESENTED SEMIGROUPS

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Let the semigroup Π be given by generators

$$a_1, a_2, \dots, a_n \quad (1)$$

and defining relations

$$A_i = B_i \quad (\text{the words } A_i \text{ and } B_i \text{ are nonempty, } i = 1, 2, \dots, k). \quad (2)$$

The set $M = \{A_i\} \cup \{B_i\}$ ($i = 1, 2, \dots, k$) is called the **basis** of the semigroup Π . The beginning (end) P of a word X will be called a **proper beginning** (**proper end**) of the word X , if $\partial(P) \geq \frac{1}{2}\partial(X)$.

We shall say that the semigroup Π with generators (1) and defining relations (2) belongs to the class $K_{1/2}$, if its basis M satisfies the following conditions: a) if $A, B \in M$, $B \stackrel{\circ}{=} RPQ$, and P is a proper beginning of the word A , then R is empty; b) if $A, B \in M$, $B \stackrel{\circ}{=} RPQ$, and Q is a proper end of the word A , then Q is empty.

Theorem 1. *If $\Pi \in K_{1/2}$, then for every word Z in the alphabet (1) there exists only a finite number of words equal to it in Π .*

From Theorem 1 it follows easily that

Theorem 2. *There exists an algorithm that solves the identity problem for all semigroups of the class $K_{1/2}$.*

The same result for a substantially narrower class of semigroups was obtained by E. I. Grindlinger (see (2)).

By $\overline{K}_{1/2}$ we denote the class of semigroups obtained analogously to the class $K_{1/2}$, if in the definition of proper beginnings and proper ends one considers, instead of the non-strict inequality, the strict inequality $\partial(P) > \frac{1}{2}\partial(X)$.

It is interesting to note that in the class $\overline{K}_{1/2}$ one can indicate an example of a finitely presented semigroup with an unsolvable identity problem. Such examples are easily constructed on the basis of the known result of G. S. Tseitin ⁽⁴⁾. In particular, the semigroup given by the generators $a, b, c, d, e, m, n, p, q, m', n', y', q', \alpha, \beta, \gamma, \delta, \eta, \theta$ and defining relations

$$\begin{aligned} ac = ca, \quad ad = da, \quad bc = cb, \quad bd = db, \quad e = mn, \quad nc = cp, \quad mc = cq, \\ qpa = e, \quad e = m'n', \quad n'd = dp', \quad m'd = dq', \quad q'p'b = e, \quad ca = \alpha\beta\gamma, \\ c\alpha = \delta\theta, \quad cc = \delta\eta, \quad e = \alpha\gamma, \quad \eta aa = \theta\beta, \end{aligned} \quad (3)$$

belongs to the class $\overline{K}_{1/2}$, and in it the identity problem is unsolvable. An analogous example is indicated in the paper of E. I. Grindlinger ⁽³⁾.

We proceed to the proof of Theorem 1. Let $\Pi \in K_{1/2}$. A word

$$E \stackrel{\circ}{=} S_1 S_2 \dots S_t \quad (4)$$

in the alphabet (1) will be called **normal** if the factors $S_1, S_2, \dots, S_t$ satisfy the conditions:

- I. Each S_i is contained in some $C_i \in M$.
- II. S_1 is a proper beginning of C_1 ; S_t is a proper end of C_t .
- III. For each $i < t$, either S_i is a proper end C_i , or S_{i+1} is a proper beginning C_{i+1} .

Every decomposition (4) with properties I, II, and III will be called a **normal decomposition** of the word E .

Lemma 1. *If the word PAQ , where $A \in M$, has a normal decomposition consisting of t factors, then one can indicate such a normal decomposition (4) of the word PAQ that $P \stackrel{\circ}{=} S_1 S_2 \dots S_{j-1}$, $A \stackrel{\circ}{=} S_j$.*

Proof. Let $PAQ \stackrel{\circ}{=} R_1 R_2 \dots R_t$ be a normal decomposition, where $C_\xi \stackrel{\circ}{=} u_\xi R_\xi q_\xi$, $C_\xi \in M$ ($\xi = 1, 2, \dots, t$). Obviously, one can indicate such an $i \leq t$ that $P \stackrel{\circ}{=} R_1 R_2 \dots R'_i$, $R_i \stackrel{\circ}{=} R'_i A'$, $A \stackrel{\circ}{=} A' A''$, where A' is nonempty. Two cases are possible: 1) $\partial(A') \geq \frac{1}{2}\partial(A)$; 2) $\partial(A') < \frac{1}{2}\partial(A)$.

Case 1. In this case A' is a proper beginning of the word A ; $C_i \stackrel{\circ}{=} u_i R'_i A' q_i$, and, by condition a), $u_i R'_i$ is empty and $R_i \stackrel{\circ}{=} A'$. If in addition A'' is empty, then $R_i \stackrel{\circ}{=} A$. Then put $S_1 \varpi R_1, \dots, S_{i-1} \varpi R_{i-1}$, $S_i \varpi A$, $S_{i+1} \varpi R_{i+1}, \dots, S_t \varpi R_t$ (ϖ denotes equality by definition). Obviously, the required decomposition (4) has been found.

Let A'' be nonempty. Then R_i is not a proper end; indeed, suppose R_i is a proper end. Then q_i is empty and $C_i \stackrel{\circ}{=} A'$, i.e. A' is a proper end C_i , and

then A' is an end of A , i.e. A'' is empty, contrary to the assumption. From condition III it follows that R_{i+1} is a proper beginning. If $A'' \stackrel{\circ}{=} R_{i+1}Z_1$ for some Z_1 , then R_{i+1} must be a beginning of A , which is impossible, since A' is nonempty. Therefore $R_{i+1} \stackrel{\circ}{=} A''R_{i+1}'$, where R_{i+1}'' is nonempty. Note that if R_{i+1} is at the same time also a proper end, then R_{i+1}'' is a proper end C_{i+1} . Put $S_1 \rightleftharpoons R_1, \dots, S_{i-1} \rightleftharpoons R_{i-1}, S_i \rightleftharpoons A, S_{i+1} \rightleftharpoons R_{i+1}'', S_{i+2} \rightleftharpoons R_{i+2}, \dots, S_t \rightleftharpoons R_t$, and the assertion of Lemma 1 is fulfilled for $j = i$.

Case 2. We have $\partial(A'') > \frac{1}{2}\partial(A)$, consequently A'' is a proper end of A . If for some Z_2 we have $R_{i+1} \stackrel{\circ}{=} A''Z_2$, then Z_2 is empty, since A'' must be an end of C_{i+1} . Therefore one may suppose that $A'' \stackrel{\circ}{=} R_{i+1}Z_3$ for some Z_3 .

Since A' is nonempty, R_{i+1} is not a proper beginning. Indeed, suppose R_{i+1} is a proper beginning. Then, since $A \stackrel{\circ}{=} A'R_{i+1}Z_3$, R_{i+1} is a beginning of A , i.e. A' is empty, contrary to the assumption. From condition III it follows that R_i is a proper end. If at the same time R_i' is empty, then $R_i \stackrel{\circ}{=} A'$, and A' must be an end of A . But this is impossible, since A'' is nonempty. Hence R_i' is nonempty. At the same time, if R_i was also a proper beginning, then $R_i \stackrel{\circ}{=} C_i$. Then R_i' is a proper beginning for C_i , since $\partial(A) < \frac{1}{2}\partial(C_i)$.

If R_{i+1} is a proper end, then Z_3 is empty. Then put $S_1 \rightleftharpoons R_1, \dots, S_{i-1} \rightleftharpoons R_{i-1}, S_i \rightleftharpoons R_i', \dots, S_{i+1} \rightleftharpoons A, S_{i+2} \rightleftharpoons R_{i+2}, \dots, S_t \rightleftharpoons R_t$, and for $j = i+1$ the assertion of Lemma 1 is fulfilled.

Let R_{i+1} not be a proper end. Then R_{i+2} is a proper beginning. If $Z_3 \stackrel{\circ}{=} R_{i+2}Z_4$ for some Z_4 , then R_{i+2} must be a beginning of A , but this is impossible. Consequently, we have $R_{i+2} \stackrel{\circ}{=} Z_3R_{i+2}'$, where R_{i+2}' is nonempty. Note that if R_{i+2} is at the same time also a proper end, then $R_{i+2} \stackrel{\circ}{=} C_{i+2}$. Then R_{i+2}' is a proper end C_{i+2} , since $\partial(Z_3) < \frac{1}{2}\partial(C_{i+2})$. The latter follows from the fact that in the word $A \stackrel{\circ}{=} A'R_{i+1}Z_3$, Z_3 is not a beginning. Put $S_1 \rightleftharpoons R_1, \dots, S_{i-1} \rightleftharpoons R_{i-1}, S_i \rightleftharpoons R_i', S_{i+1} \rightleftharpoons A, S_{i+2} \rightleftharpoons R_{i+2}', S_{i+3} \rightleftharpoons R_{i+3}, \dots, S_t \rightleftharpoons R_t$, $j = i+1$. Lemma 1 is proved.

From Lemma 1 it follows immediately that

Lemma 2. *If PAQ has a normal decomposition into t factors, and $PAQ \rightarrow PBQ$ is an elementary transformation of the semigroup Π (see (1), p. 7), then PBQ also has a normal decomposition into t factors.*

Let us call a normal word V the **maximal normal beginning** of the word VQ , if no word VQ' , with nonempty Q_1 and $Q \stackrel{\circ}{=} Q_1Q_2$

is not a normal word. The following lemma follows directly from Lemma 2.

Lemma 3. *Let V be the maximal normal beginning of the word VQ , and let $V \rightarrow V_1$ be an elementary transformation of Π . Then V_1 is the maximal normal beginning of the word V_1Q .*

Lemma 4. *If V is the maximal normal beginning of the word VY and $VY \stackrel{\circ}{=} VY'$*

PAQ , where $A \in M$, then either $\partial(V) < \partial(P)$, or $\partial(V) \geq \partial(PA)$.

Proof. Suppose that $\partial(P) \leq \partial(V) < \partial(PA)$. Then we have $V \overset{\circ}{=} PQ'$, $A \overset{\circ}{=} A'A''$, $Y \overset{\circ}{=} A''Q$, where A'' is nonempty. We shall prove that VA'' is a normal word. Let $R_1 \dots R_t$ be the normal decomposition of the word V . It is enough to prove that A'' is a proper end. Suppose this is not so. Then A' is a proper beginning.

Either R_t is an end of the word A' , or A' is an end of R_t . Arguing as in the proof of Lemma 1, it is easy to verify that both these cases are impossible. Consequently, VA'' is a normal beginning of VY . This contradicts the fact that V is the maximal normal beginning of VY , since A'' is nonempty. Lemma 4 is proved.

Lemma 5. If $\partial(X) = l$ and $X = Y$ in Π , then $\partial(Y) \leq lm^l$, where $m = \max\{\partial(A_i), \partial(B_i)\}$.

We shall prove this assertion by induction on the length of the word X .

Let $\partial(X) = 0$, i.e., X is the empty word. If $X = Y$ in Π , then Y is the empty word, since the defining relations (2) consist of nonempty words. Therefore, $\partial(Y) = 0$.

Let $\partial(X) = l > 0$. Then $X \overset{\circ}{=} aW$, where a is one of the generators (1). By the induction hypothesis, the length of any word W_j equal to W in Π does not exceed $(l-1)m^{l-1}$. Let $Y \overset{\circ}{=} aW'$. Among all sequences of elementary transformations $aW_j \overset{\circ}{=} F_0 \rightarrow \dots \rightarrow F_q \overset{\circ}{=} Y$, for various words W_j equal to W , choose a sequence

$$aW_k \overset{\circ}{=} E_0 \rightarrow E_1 \rightarrow \dots \rightarrow E'_i \rightarrow E_{i+1} \rightarrow \dots \rightarrow E_2 \overset{\circ}{=} Y, \quad (5)$$

containing the smallest number of transformations. Let $aW_k \overset{\circ}{=} VQ$, where V is the maximal normal beginning of VQ .

If V is empty, then the first elementary transformation in (5) has the form $P_1AP_2 \rightarrow P_1BP_2$, where P_1 is nonempty. Then $P_1 \overset{\circ}{=} aP'_1$. Put $W_s \overset{\circ}{=} P'_1AP_2$. Obviously $W_s = W_k = W$ in Π , and the sequence $aW_s \overset{\circ}{=} E_1 \rightarrow E_2 \rightarrow \dots \rightarrow E_r \overset{\circ}{=} Y$ is shorter than (5), which contradicts the condition by which the sequence (5) was chosen. Consequently, if V is empty, then $Y \overset{\circ}{=} aW_k$, and

$$\partial(Y) \leq 1 + (l-1)m^{l-1} \leq lm^l.$$

If V is nonempty, then $V \overset{\circ}{=} aV'$ and $W_k \overset{\circ}{=} V'Z$. By induction on the number i of the word E_i , we shall prove that $E_i \overset{\circ}{=} V_iZ$, where V_i is the maximal normal beginning of E_i , and that there exists a sequence

$$aV' \overset{\circ}{=} V_0 \rightarrow V_1 \rightarrow \dots \rightarrow V_i, \quad (6)$$

containing exactly i elementary transformations. The word E_0 has the indicated form. Suppose that our assertion is true for E_t , and let us prove it for E_{t+1} . Let the elementary transformation $E_t \rightarrow E_{t+1}$ have the form $PAQ \rightarrow PBQ$, where $A, B \in M$. By Lemma 4, either $\partial(V_t) < \partial(P)$, or $\partial(V_t) \geq \partial(PA)$. We shall prove that the first case is impossible. Indeed, in this case we have $P \stackrel{\circ}{=} V_t P'$ and $Z \stackrel{\circ}{=} P' A Q$. Then $W_k \stackrel{\circ}{=} V' P' B Q$ in Π , whence $V' P' B Q = W$ in Π . With the aid of the sequence (6), it is easy to construct the sequence of elementary transformations

$$aV'P'BQ \rightarrow V_1P'BQ \rightarrow \dots \rightarrow V_tP'BQ \stackrel{\circ}{=} E_{t+1} \rightarrow E_{t+2} \rightarrow \dots \rightarrow E_r,$$

which is shorter than the sequence (5).

In the second case we have $V_t \stackrel{\circ}{=} PAV'_t$ and $Q \stackrel{\circ}{=} V'_{tZ}$. Then $E_{t+1} \stackrel{\circ}{=} PBV'_{tZ}$, and the transition $PAV'_t \rightarrow PBV'_t$ is an elementary transformation of Π . By Lemma 3, PBV'_t is the maximal normal beginning of the word PBV'_tZ , i.e. PBV'_t is the required maximal normal beginning of V_{t+1} . Consequently, $Y \circ V_{rZ}$, where $V_r = V$ in Π . Then

$$\begin{aligned} \partial(Y) &= \partial(V_r) + \partial(Z) \leq m[1 + (l-1)m^{l-1} - \partial(Z)] + \partial(Z) = \\ &= \partial(Z)(1-m) + lm^l + m(1-m)^{l-1} \leq lm^l. \end{aligned}$$

Lemma 5 is proved.

The assertion of Theorem 1 follows directly from Lemma 5.

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References Cited

- ¹ S. I. Adyan, Tr. Mat. Inst. im. V. A. Steklova, Academy of Sciences of the USSR, **85** (1966).
- ² E. I. Grinlinger, Siberian Math. J., **5**, No. 1, 77–85 (1964).
- ³ E. I. Grinlinger, DAN, **171**, No. 3, 519 (1966).
- ⁴ G. S. Tseitlin, Tr. Mat. Inst. im. V. A. Steklova, Academy of Sciences of the USSR, **52**, 172 (1958).

Note: Figure translations are in progress. See original paper for figures.

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