

EXTENSION OF RECURSIVE HIERARCHIES AND $\backslash(R\backslash)$ -OPERATIONS

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Abstract

Full Text

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MATHEMATICS

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EXTENSION OF RECURSIVE HIERARCHIES AND R -OPERATIONS

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Here we consider extensions of recursive hierarchies that arise as a result of going beyond the constructive second class of ordinal numbers. For example, the $\mathfrak{H}$ -hierarchy*, whose finite levels were defined in ⁽⁸⁾, was extended in ⁽²⁾ on the basis of the constructive first, second, and third number classes and continued still further in ⁽¹⁰⁾ on the basis of a constructive version of higher classes of ordinal numbers. To estimate such extensions of recursive hierarchies we apply the R -operations* of A. N. Kolmogorov, which makes it possible to obtain estimates more precise than in ⁽¹⁰⁾ and ⁽⁴⁾. Analogously to the finite levels of the classical hierarchy of R -sets ⁽¹¹⁾, we define a sequence of classes $\mathfrak{R}_n$ ($n = 0, 1, 2, \dots$) of sets of natural numbers and establish (where $C\mathfrak{R}_n$ denotes the class of complements of sets of the class $\mathfrak{R}_n$): 1) the $\mathfrak{H}$ -hierarchy ⁽¹⁰⁾ and the H_h -hierarchy ⁽⁴⁾ do not go beyond the class $\mathfrak{R}_2 \cap C\mathfrak{R}_2^{**}$; 2) in each of the classes $\mathfrak{R}_{n+1} \cap C\mathfrak{R}_{n+1}$ there is a transfinite hierarchy of sets not belonging to $\mathfrak{R}_n \cup C\mathfrak{R}_n$; 3) all $\mathfrak{R}_n \subset \Sigma_2^1 \cap \Pi_2^1$ (in ⁽⁴⁾, ⁽¹⁰⁾ the class $\Sigma_2^1 \cap \Pi_2^{1*}$ is indicated as an estimate for the sets covered by the $\mathfrak{H}$ - and H_h -hierarchies).

1. R -operations. $N = \{0, 1, 2, \dots\}$; a, b, i, m, n are variables on N ; N^N is the set of all functions $f : N \rightarrow N$; $\mathcal{P}N$ is the set of all subsets of N ; ξ, η are variables on $\mathcal{P}N$. Subsets of $\mathcal{P}N$ will be called **bases**. The δs -operation Φ_M with base M is the function which, to each sequence $\{E_i\}_{i \in N}$ of sets of an arbitrary space E (in our case always $E = N$), assigns the set

$$\Phi_M\{E_i\}_i = \{x : (\exists \xi \in M)(\forall i \in \xi)[x \in E_i]\}.$$

The **completion** of a base M is called the base

$$\widetilde{M} = \{\xi : (\exists \eta \in M)[\eta \subseteq \xi]\}.$$

For equality of the δs -operations Φ_{M_1} and Φ_{M_2} it is necessary and sufficient that $\widetilde{M}_1 = \widetilde{M}_2$.

For any base M , let

$$M^c = \{\xi : (N \setminus \xi) \notin M\}$$

and $M' = (\widetilde{M})^c$; then

$$\Phi_{M'}\{E_i\} = \{x : (\forall \xi \in M)(\exists i \in \xi)(x \in E_i)\}.$$

Let all tuples (i.e., finite ordered sets) of natural numbers be enumerated one-to-one by all natural numbers, with 0 the number of the empty tuple, and $[m_0, \dots, m_i]$ the number of the tuple $m_0, \dots, m_i$. Let

$$\theta(0, n) = [n]$$

and

$$\theta(m, n) = [m_0, \dots, m_i, n] \quad \text{for } m = [m_0, \dots, m_i];$$

we assume the enumeration of tuples is such that $\theta(m, n)$ is a general recursive function (g.r.f.). Let

$$\theta_m(n) = \theta(m, n)$$

and

$$\theta_m^{-1}(\xi) = \{n : \theta_m(n) \in \xi\}.$$

For any base M define the base $\mathbf{R}(M)$ as follows:

$$\mathbf{R}(M) = \{\xi : 0 \in \xi \text{ \& } (\forall m \in \xi)[\theta_m^{-1}(\xi) \in M]\}.$$

Obviously,

$$\widetilde{M}_1 = \widetilde{M}_2 \Rightarrow \widetilde{\mathbf{R}}(M_1) = \widetilde{\mathbf{R}}(M_2)$$

($\widetilde{\mathbf{R}}(M)$ is the completion of $\mathbf{R}(M)$). The δs -operation Φ_K with base $K = \mathbf{R}(M)$ is called the **R -operation over Φ_M** . (This definition corresponds to ⁽¹¹⁾.)

* The definition is given below.

** This also clarifies the connection between the $\mathfrak{H}$ -hierarchy (extended as in ⁽¹⁰⁾) and the effective R -hierarchy; on this connection see ⁽²⁾.

For any function $F : \mathcal{P}N \rightarrow \mathcal{P}N$ define the function $\mathbf{L}_F : \mathcal{P}N \rightarrow \mathcal{P}N$ as follows. First, for any $\xi \subseteq N$ and ordinal $\nu < \Omega$ (Ω is the least uncountable ordinal) define ξ^ν : 1) $\xi^0 = \xi$; 2) $\xi^{\nu+1} = F(\xi^\nu)$; 3) if ν is a limit ordinal, then $\xi^\nu = \bigcap_{\mu < \nu} \xi^\mu$; then put

$$\mathbf{L}_F(\xi) = \bigcap_{\Omega < \nu} \xi^\nu.$$

Lemma (adapted from ⁽¹²⁾). Let $M \subseteq \mathcal{P}N$ and

$$F(\xi) = \{m : m \in \xi \& \theta_m^{-1}(\xi) \in M\}.$$

Then

$$\widetilde{\mathbf{R}}(M) = \{\xi : 0 \in \mathbf{L}_F(\xi)\}.$$

If $X \subseteq \mathcal{P}N$ (or $X \subseteq N$), then let, in accordance with (1), $X \in \Sigma_n^1 \cap \Pi_n^1$ mean that the predicate $\xi \in X$ (or, respectively, $i \in X$) is expressible in both n -functional-quantifier Kleene forms ⁽⁷⁾.

Theorem 1. For any base M , if $\widetilde{M} \in \Sigma_2^1 \cap \Pi_2^1$, then also

$$\widetilde{\mathbf{R}}(M) \in \Sigma_2^1 \cap \Pi_2^1.$$

The Σ_2^1 -form for $\widetilde{\mathbf{R}}(M)$ is obtained from the definition of $\mathbf{R}(M)$, and the Π_2^1 -form from the lemma and Patnam's method ⁽¹³⁾. (If the classes of Σ_2^1 - and Π_2^1 -subsets of $\mathcal{P}N$ are numbered, for example, as in ⁽⁷⁾, then one can find such general recursive functions $g_0(m, n)$ and $g_1(m, n)$ that, for any pair m, n of numbers of the two forms of the base $\widetilde{M}$, the numbers $g_0(m, n)$ and $g_1(m, n)$ are numbers of the two forms of the base $\widetilde{\mathbf{R}}(M)$.)

Define a sequence of bases R_n ($n = 0, 1, 2, \dots$) as follows: $R_0 = \{N\}$ (i.e. R_0 contains only one element, N), $R_{n+1} = \mathbf{R}(R_n)$.

The δs -operation Φ_{R_n} with base R_n is called an R^n -operation. Thus, an R^0 -operation is the operation of intersection

$$\Phi_{R_0}\{E_i\} = \bigcap_{i \in N} E_i,$$

and an R^{n+1} -operation is an R -operation over Φ'_{R_n} . From Theorem 1 we obtain

Corollary*. For any $n \in N$,

$$\widetilde{R}_n \in \Sigma_2^1 \cap \Pi_2^1 \quad \text{and} \quad \widetilde{R}'_n \in \Sigma_2^1 \cap \Pi_2^1.$$

2. The classes $\mathfrak{R}_n$. A one-place partial recursive function (p.r.f.) with Gödel number (see ⁽⁶⁾) a is denoted by $\langle a \rangle$. By τ we denote the family (see ⁽³⁾) of all recursively enumerable subsets of N , having the following numbering: for any $a \in N$, the member of the family τ with number a , τa , is the domain of definition of the p.r.f. $\langle a \rangle$.

The class of all sets $\xi (\subseteq N)$ representable in the form

$$\xi = \{m : (\exists \eta \in R_n)(\forall i \in \eta)P(m, i)\},$$

where $P(m, i)$ is some recursively enumerable predicate, will be called the class $\mathfrak{R}_n$ ($n = 0, 1, 2, \dots$). $C\mathfrak{R}_n$ is the class of complements (with respect to N) of sets of the class $\mathfrak{R}_n$.

Obviously, the class $\mathfrak{R}_n$ consists of all sets obtained by an R^n -operation over all possible recursively enumerable sequences of members of the family τ . We provide each of the classes $\mathfrak{R}_n$ with a certain numbering, defining the families ρ_n as follows:

$$\rho_n = H_{\widetilde{R}_n} \tau,$$

where $H_{\widetilde{R}_n}$ is the standard operator with base $\widetilde{R}_n$ (⁽³⁾). Since any δs -operation Φ_M with base M coincides with the set-theoretic operation $\Psi_{\widetilde{M}}$ with base $\widetilde{M}$, then, by the definition of the standard operator, the member of the family ρ_n with number a ,

$$\rho_n a = \Phi_{R_n} \{E_i\},$$

where

$$E_i = \begin{cases} \tau \langle a \rangle(i), & \text{for those } i \in N \text{ for which } \langle a \rangle(i) \text{ is defined,} \\ \emptyset, & \text{for the remaining } i \in N. \end{cases}$$

The families $\bar{\rho}_n$ are defined as follows: for any $a \in N$,

$$\bar{\rho}_n a = N \setminus \rho_n a.$$

Obviously, the totality of all members of the family ρ_n coincides with $\mathfrak{R}_n$, and the totality of all members of $\bar{\rho}_n$ with $C\mathfrak{R}_n$.

Theorem 2. For any $n \in N$,

$$\mathfrak{R}_n \subset \mathfrak{R}_{n+1}, \quad C\mathfrak{R}_n \subset \mathfrak{R}_{n+1} \quad \text{and} \quad \mathfrak{R}_n \neq C\mathfrak{R}_n.$$

Proof by means of the properties of R -operations established in (⁽⁵⁾).

* Cf. ⁽¹¹⁾, Corollary 2 of Theorem 12. See also our remark after Theorem 3.

Theorem 3. For all $n \in N$, $\mathfrak{R}_n \subset \Sigma_2^1 \cap \Pi_2^1$.

Proof on the basis of the corollary to Theorem 1.

Remark. With the aid of the system of notations for ordinals of the first two constructive number classes ⁽⁹⁾, the sequence of bases R_n and classes $\mathfrak{R}_n$ can be continued transfinitely, obtaining an effective analogue of the classical hierarchy of R -sets ⁽¹¹⁾. As the basic space, instead of N one may consider the space

$$N^{m,n} = (N^N)^m \times N^n \quad (m, n = 0, 1, \dots; m + n > 0) \quad (1),$$

taking the family of all effectively open sets of this space instead of τ . The corollary of Theorem 1 and Theorems 2 and 3 can be extended to both these cases.

3. Hierarchies in the class $\mathfrak{R}_2$. Introduce the one-one recursive functions $\sigma(m, n)$, $(n)'$, and $(n)''$, establishing a one-to-one correspondence between N^2 and N . The function

$$\text{hj} : \mathcal{P}N \rightarrow \mathcal{P}N,$$

which assigns to each $\xi \subseteq N$ the set

$$\text{hj}(\xi) = \{m : (\forall f) (\mathfrak{R}_n) T_1^{\xi, f}(m, m, n)\},$$

is called a **hyperjump** (hyperjump, ^(4,13)). The hyperjump can be represented by means of the δs -operation $\Phi_{R_1'}$; with the aid of this representation and known properties of R -operations ^(5,12) one obtains

Theorem 4. If $\xi \in \mathfrak{R}_2 \cap C\mathfrak{R}_2$, then also $\text{hj}(\xi) \in \mathfrak{R}_2 \cap C\mathfrak{R}_2$, and there exists a recursive function $g(m, n)$ such that, if

$$\xi = \rho_2 m = \rho_2 n,$$

then

$$\text{hj}(\xi) = \rho_2(g(m, n))' = \rho_2(g(m, n))''.$$

The field of a binary relation $P(m, n)$ is the set

$$\{m : (\exists n)[P(m, n) \vee P(n, m)]\}.$$

An irreflexive transitive binary relation $m <_W n$ with field $W (\subseteq N)$ is called a **system of notations** (s.o.), if: a) $1 \in W$ and

$$(\forall n \in W)[n \neq 1 \Rightarrow 1 <_W n];$$

b) every nonempty subset of the field W has a $<_W$ -minimal element.

For any function $F : \mathcal{P}N \rightarrow \mathcal{P}N$ and s.o. $<_W$ with field W , denote by $\Pi(F, <_W)$ the class of all and only those functions $\mathfrak{F} : W \rightarrow \mathcal{P}N$ for each of which there exist partial recursive functions $p(a)$, $q(a, b)$, and $r(a, b)$ such that, if $a \in W$ and $a \neq 1$, then:

a) $p(a) <_W a$ and

$$\{b : q(a, b) \in \mathfrak{F}(p(a))\} = \{b <_W a\};$$

b)

$$\mathfrak{F}(a) = \{m : r(a, m) \in F(U_a)\},$$

where

$$U_a = \{\sigma(b, m) : q(a, b) \in \mathfrak{F}(p(a)) \text{ \& } m \in \mathfrak{F}(b)\}.$$

Thus, if $\mathfrak{F} \in \Pi(F, <_W)$, then the value $\mathfrak{F}(a)$, for $a \in W$ and $a \neq 1$, is determined according to the scheme given in b), through the values $\mathfrak{F}(b)$ for $b <_W a$.

Theorem 5. If $\mathfrak{F} \in \Pi(\text{hj}, <_W)$, where $<_W$ is an s.o. with field W , and if

$$\mathfrak{F}(1) \in \mathfrak{R}_2 \cap C\mathfrak{R}_2,$$

then there exists a recursive function $f(a)$ such that for any $a \in W$

$$\mathfrak{F}(a) = \rho_2(f(a))' = \rho_2(f(a))''.$$

Proof on the basis of the recursion lemma ^(4,10) and Theorem 4.

Corollary. Let an s.o. $<_h$ with field D_h and a function

$$H_h : D_h \rightarrow \mathcal{P}N$$

be defined by ⁽⁴⁾. Then

$$H_h(x) \in \mathfrak{R}_2 \cap C\mathfrak{R}_2$$

for all $x \in D_h$.

Now consider the transfinite iteration of the hyperjump on the basis of the constructive version of the class of all attainable ordinals. Let an s.o. $<_{\widetilde{C}}$ with field $\widetilde{C}$, and for each $a \in \widetilde{C}$ subsystems I_a ($I_a \subset <_{\widetilde{C}}$) with field $\text{dom } I_a$ (interpreted as the constructive number class of index a) be defined as in ⁽¹⁰⁾. For each $a \in \widetilde{C}$ there is in $\widetilde{C}$ a unique element immediately following a , 2^a . If $a \in \widetilde{C}$ and

$$(\forall n)[a \neq 2^n],$$

then $a = 3^b 5^n$, where $b <_{\widetilde{C}} a$ and n is the Gödel number of such a partial recursive function that

$$(\forall i \in \text{dom } I_b)[\langle n \rangle(i) <_{\widetilde{C}} a].$$

Theorem 6. There exists a recursive function $g(a)$ such that for any $a \in \widetilde{C}$

$$\sigma(I_a) = \rho_2(g(a))' = \rho_2(g(a))''.$$

Proof by means of Theorem 5.

For all $a \in \widetilde{C}$, the sets $\mathfrak{H}_a (\subseteq N)$ are defined as follows ⁽¹⁰⁾: 1) $\mathfrak{H}_1 = N$; 2) $\mathfrak{H}_{2^a} = \text{hj}(\mathfrak{H}_a)$, where $a \in \widetilde{C}$; 3) if $a \in \widetilde{C}$ and $a = 3^b 5^n$, then

$$\mathfrak{H}_a = \{m : (m)_1 \in \text{dom } I_b \text{ \& } (m)_0 \in \mathfrak{H}_{\langle n \rangle \langle (m)_1 \rangle}\}.*$$

* For $m > 0$, $(m)_i$ is the exponent of the i -th prime number in the representation of m as a product of powers of primes; $(0)_i = 0$ (see ⁽⁶⁾).

From Theorem 6, with the aid of Theorem 4, we obtain

Corollary. For all $a \in \widehat{C}$, $\mathfrak{H}_a \in \mathfrak{R}_2 \cap C\mathfrak{R}_2$.

4. Hierarchies in the classes $\mathfrak{R}_{n+1}$. Theorem 5 can be generalized as follows.

Theorem 7. For any $n = 0, 1, 2, \dots$: if $\mathfrak{F} \in \Pi(F, <_W)$, where $<_W$ is a c.o. with field W , and the function $F : \mathfrak{P}N \rightarrow \mathfrak{P}N$ is representable in the form

$$F(\xi) = \{m : (\forall \eta \in R_n)(\exists i \in \eta) P^{\xi, \eta}(m, i)\}$$

with some general-recursive $P^{\xi, \eta}$, and if $\mathfrak{F}(1) \in \mathfrak{R}_{n+1} \cap C\mathfrak{R}_{n+1}$, then there exists an o.r.f. $f(a)$ such that, for every $a \in W$,

$$\mathfrak{F}(a) = \rho_{n+1}(f(a))' = \bar{\rho}_{n+1}(f(a))''.$$

On the basis of this theorem, in each of the classes $\mathfrak{R}_{n+1} \cap C\mathfrak{R}_{n+1}$ it is easy to indicate (taking, for example, $P^{\xi, \eta}(m, i) \iff T_1^{\xi, \eta}(m, m, i)$ and, as the c.o. $<_W$, the Kleene system $<_0$ ⁽⁹⁾) a transfinite sequence of sets that do not belong to $\mathfrak{R}_n \cup C\mathfrak{R}_n$ and have strictly increasing degrees of recursive unsolvability.

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Note: Figure translations are in progress. See original paper for figures.

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