

ON THE THEORY OF CONVERGENCE OF APPROXIMATE SOLUTIONS OF AN OPERATOR EQUATION

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Abstract

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MATHEMATICS

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ON THE THEORY OF CONVERGENCE OF APPROXIMATE SOLUTIONS OF AN OPERATOR EQUATION

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Recently many works have appeared on the study of the convergence of approximate solutions of ill-posed problems (¹⁻⁵). In the present note essentially the same questions are considered, but under different initial assumptions.

Consider in a Hilbert space H the linear equation

$$A\varphi = f \in H, \quad A = A^*. \quad (1)$$

Let $e_1, e_2, \dots, e_n, \dots$ be an orthonormal system of eigenvectors of the operator A , with $Ae_i = 0$, $(f, e_i) = 0$ for $i = 1, 2, \dots, n$. We associate with equation (1) the equation

$$A_h\varphi_h = f_h \in H, \quad h > 0, \quad (2)$$

where the linear operator A_h acts in the same Hilbert space H ,

$$\|f - f_h\| = O(h^\alpha), \quad \alpha > 0, \quad (3)$$

and on the solutions of equation (1) corresponding to $f \in H_1 \subset H$, the equalities

$$\|(A - A_h)\varphi\| = O(h^\beta), \quad \beta > 0; \quad (4)$$

$$(A_h\varphi, e_i) = O(h^\varepsilon), \quad i = 1, 2, \dots, n, \quad \varepsilon > 0. \quad (5)$$

hold. Suppose, moreover, that

$$\|A_h^{-1}\| = O(h^{-\gamma}), \quad \gamma \geq 0; \quad (6)$$

$$(f_h, e_i) = O(h^{\varepsilon_1}), \quad i = 1, 2, \dots, n, \quad \varepsilon_1 > 0. \quad (7)$$

We represent the solution φ_h of equation (2) in the form

$$\varphi_h = \varphi_0 + \varphi_1 + \varphi_2. \quad (8)$$

Here φ_0 is a solution of equation (1), and φ_1 is a solution of the equation

$$A\varphi_1 = -f_1, \quad (9)$$

where f_1 satisfies the equality

$$A_h\varphi_0 - f_h = f_1 + \tilde{f}_1, \quad (10)$$

$$\|\tilde{f}_1\| = O(h^\mu), \quad \mu = \min(\varepsilon, \varepsilon_1).$$

From conditions (5), (7) it follows that the solution of equation (9) exists for $h^{-\nu}f_1 \in H_1$, and, by virtue of relations (3), (4),

$$\|f_1\| = O(h^\nu), \quad \nu = \min(\alpha, \beta) \quad (11)$$

and, consequently, for $h^{-\nu}f_1 \in H_1$

$$\|\varphi_1\| = O(h^\nu). \quad (12)$$

Substituting (8) into equation (2), by virtue of the equalities (4), (6), (10), (11), (12), we obtain

$$\begin{aligned} \|A_h\varphi_2\| &= \|f_h - A_h\varphi_0 - A_h\varphi_1\| = \|-f_1 - \tilde{f}_1 - A_h\varphi_1 + A\varphi_1 - A\varphi_1\| \leq \\ &\leq \|\tilde{f}_1\| + \|(A - A_h)\varphi_1\| + \|(A\varphi_1 + f_1)\| = O(h^{\min(\mu, 2\nu)}), \end{aligned} \quad (13)$$

$$\|\varphi_2\| = O(h^{\min(\mu, 2\nu) - \gamma}). \quad (14)$$

From (8), (12), (14) it follows that

$$\|\varphi_h - \varphi_0\| = O(h^{\min[\nu, \min(\mu, 2\nu) - \gamma]}). \quad (15)$$

Thus, we have proved the following

Theorem (convergence criterion). If $f, h^{-\nu}f_1 \in H_1$,

$$\lim_{h \rightarrow 0} \|\varphi_0 - \varphi_h\| = 0,$$

if

$$\mu = \min(\varepsilon, \varepsilon_1) > \min(\alpha, \beta) = \nu, \quad \min(\mu, 2\nu) > \gamma.$$

Remark. It follows from relation (15) that the larger the instability exponent γ , the larger the approximation exponents μ and ν must be.

Example. As an example, let us consider the system of two linear algebraic equations

$$A\varphi = f = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \quad (16)$$

It is obvious that the eigenvectors of the matrix of the system of equations (16) are

$$y_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad y_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (17)$$

We approximate equation (16) by the equation

$$A_h \varphi_h = f_h = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad A_h = \begin{pmatrix} 1+h+h^2 & 1 \\ 1 & 1+h \end{pmatrix}. \quad (18)$$

As a solution of the system (16) we take

$$\varphi_0 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}. \quad (19)$$

If H_1 is the set of vectors of the form

$$x = \begin{pmatrix} a \\ a \end{pmatrix}, \quad |a| < a_0 = \text{const} < \infty, \quad (20)$$

then $f, h^{-1}f_1 \in H_1$ and $\mu = 2, \nu = 1, \gamma = 1, \min(\mu, 2\nu) = 2$. Consequently, according to the theorem, the solution of the system of equations (18) converges as $h \rightarrow 0$ to the solution (19) of equation (16).

The scheme of reasoning presented above is readily applied to the proof of convergence theorems also for evolution equations with unbounded operators.

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Note: Figure translations are in progress. See original paper for figures.

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