

ON CLASSIFYING SPACES FOR PRINCIPAL BUNDLES WITH HAUSDORFF BASES

I. M. Gel'fand, D. B. Fuchs

1968

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196801.03582>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 513.836

MATHEMATICS

Corresponding Member of the Academy of Sciences of the USSR I. M. GELFAND, D. B. FUKS

ON CLASSIFYING SPACES FOR PRINCIPAL BUNDLES WITH HAUSDORFF BASES

In our recent work ⁽¹⁾ (see also ^(2,3)) we defined an analogue of the universal bundle for principal G -bundles with Hausdorff bases, where G is a closed Lie subgroup of the group $GL(n, \mathbf{R})$. The present note contains a generalization of this construction, making it possible to construct a universal G -bundle for any topological group G . This construction contains as special cases both the construction from article ⁽¹⁾ (with a slight modification) and an analogue of the well-known construction of Milnor ⁽⁶⁾. With its help one can extend the results of article ⁽¹⁾ to the case of an arbitrary connected Lie group and establish, for any paracompact group G , the coincidence of the group $H_{alg}^q(G; V)$ of characteristic classes of principal G -bundles (with Hausdorff bases) with coefficients in a topological G -module V and the group $H_c^q(G; V)$ of “continuous cohomology of Eilenberg–Mac Lane” ⁽⁴⁾.

1. Let G be a topological group. By a principal G -bundle we mean a triple consisting of a topological space E , on which the group G acts on the right without fixed points (i.e., if for some $g \in G$, $y \in E$ the equality $yg = y$ holds, then $g = e$), of the quotient space X , and of the projection $p : E \rightarrow X$. The principal G -bundle $\xi = (E, p, X)$ is called locally trivial in the classical sense (or simply locally trivial) if every point $x \in X$ has a neighborhood U such that the full inverse image $p^{-1}(U) \subset E$, as a G -space, is isomorphic to the product $U \times G$.

By an elementary G -object $\mathbf{T}$ we shall mean a triple $\mathbf{T} = (T, A, \mu)$, where T is a Hausdorff space with a distinguished point $*$; A is a topological semigroup with identity, containing the group G ; $\mu : T \times A \rightarrow T$ is a continuous action of the semigroup A on T such that:

- 1°. Every transformation $a \in A$ of the space T leaves the point $*$ fixed.
- 2°. Distinct elements of the semigroup A define distinct transformations of the space T .
- 3°. There exists a homotopy $\varphi_t : T \rightarrow T$ such that $\varphi_t \in A$ for all t , φ_0 is the identity transformation, and $\varphi_1(z) = *$ for all $z \in T$.

4°. There exist a neighborhood W of the group G in the semigroup A and a mapping $\lambda : W \rightarrow G$ such that: (a) every element of W is invertible in the semigroup A ; (b) $ag \in W$ for all $a \in W$, $g \in G$; (c) $\lambda(ag) = \lambda(a)g$ for all $a \in A$, $g \in G$; (d) $\lambda(g) = g$ for every $g \in G$.

Let us note that the group G acts both on the space T and on the semigroup A (by right multiplications); moreover, the invertible elements of the semigroup A form a G -invariant subset on which the group A acts without fixed points.

The principal examples of elementary objects for us will be the following two.

Example 1. G is a closed Lie subgroup of the group $GL(n, \mathbf{R})$, $T = \mathbf{R}^n$, and A is the semigroup of all linear mappings of $\mathbf{R}^n$ into itself.

Example 2. G is any topological group, T is the cone CG over G (i.e., the set of pairs (g, t) , where $g \in G$, t is a nonnegative number, $\dots$

whereby the pairs $(g', 0)$ and $(g'', 0)$ are identified for any $g', g'' \in G$. The semigroup A , as a topological space, coincides with T ; multiplication in A (and the action of A on T) is defined by the formula

$$(g_1, t_1)(g_2, t_2) = (g_1 g_2, t_1, t_2).$$

The embedding $G \subset A$ sends an element $g \in G$ to the point $(g, 1) \in A$. The element φ_t is chosen to be equal to $(e, 1 - t)$.

Fix a group G and an elementary G -object $T = (T, A, \mu)$. With every principal G -bundle $\xi = (E, p, X)$ one can associate the bundle $\xi_T(E_T, p_T, X)$ with fiber T , by putting $E_T = (E \times T)/G$ (the action of the group G on $E \times T$ is defined coordinatewise). To each element $y \in E$ there corresponds a map $\eta_y : T \rightarrow E_T$ (the composition $T = y \times T \subset E \times T \rightarrow E_T$), homeomorphically mapping T onto the fiber of the bundle ξ_T over the point $p(y)$. Obviously, $\eta_{yg} = \eta_y g$ for all $y \in E$, $g \in G$, and distinct points $y_1, y_2 \in E$ correspond to distinct maps η_{y_1}, η_{y_2} . It is also clear that the map η_y depends continuously on the point y .

Definition. A principal G -bundle $\xi = (E, p, X)$ is called **locally T -trivial** if for every point $x \in X$ there exists a continuous map $\pi_x : E_T \rightarrow T$ such that for all $y \in E$ the map $\pi_x \eta_y : T \rightarrow T$ belongs to the semigroup A , and for $py = x$ is its invertible element.

Remark. The definition would not change if we required that, for $py = x$, the element $\pi_x \eta_y \in A$ belong to $G \subset A$: to satisfy this condition it suffices to take, instead of the map π_x , the map $(\pi_x \eta_y)^{-1} \pi_x$, where $py = x$.

Obviously, the bundle induced by a locally T -trivial bundle $\xi = (E, p, X)$ under any map $f : X' \rightarrow X$ is also locally T -trivial.

Proposition 1. *Every locally T -trivial bundle is locally trivial (in the classical sense).*

Proof. Let x be an arbitrary point of the base of the locally T -trivial bundle $\xi = (E, p, X)$. As was noted, we may assume that the map $\pi_x \eta_y$ is an element

of the group G for $py = x$. The set of those points $y \in E$ for which $\pi_x \eta_y \in W$ is open in E . Since $\pi_{yg} = \eta_y g$, this set is invariant under the action of the group G on E , i.e. it is the full inverse image of some open set $U \subset X$ under the projection p . The map

$$p^{-1}(U) \rightarrow U \times G,$$

sending an element $y \in p^{-1}(U)$ to the element $(p(y), \lambda(\pi_x \eta_y)) \in U \times G$, is compatible with the action of the group G . Consequently, the G -spaces $p^{-1}(U)$ and $U \times G$ are isomorphic.

Proposition 2. *If the base X of a locally trivial bundle ξ is completely regular, then ξ is locally T -trivial.*

Proof. Let $x \in X$ be a point; $\sigma : U \rightarrow E$ a section of the surface of the bundle ξ over a neighborhood $U \ni x$; h a continuous real-valued function on X , taking values from 0 to 1, equal to 0 at the point x and to 1 outside U . Put

$$\pi_x(z) = \varphi_{h(z)} \eta_{\sigma(p_T(z))}^{-1}(z)$$

for $p_T(z) \in U$, and

$$\pi_x(z) = *$$

for the remaining z .

2. We proceed to describe the construction of the universal bundle. For a topological space X and a set I , denote by X^I the topological space whose points are families $\{x_i\}$ of points of the space X , indexed by the elements of the set I . A basis of open sets of the space X^I is formed by the sets

$$\Gamma(i_1, \dots, i_k; U_1, \dots, U_k),$$

where $i_1, \dots, i_k \in I$, $U_1, \dots, U_k$ are open sets in X ,

$$\Gamma(i_1, \dots, i_k; U_1, \dots, U_k) = \{\{x_i\} \mid x_{i_1} \in U_1, \dots, x_{i_k} \in U_k\}.$$

Denote by $\mathcal{E}_G^I$ the subspace of the space A^I consisting of those families $\{a_i\}$ in which at least one element a_i is invertible in the semigroup A . We define the action of the group G on $\mathcal{E}_G^I$ by the formula $\{a_i\}g = \{a_i g\}$. The quotient space $\mathcal{E}_G^I/G$ will be denoted by $\mathcal{B}_G^I$, and the projection $\mathcal{E}_G^I \rightarrow \mathcal{B}_G^I$ by p_G^I , or, for short, by p .

Proposition 3. *The space $\mathcal{B}_G^I$ is Hausdorff.*

Proof is obvious.

Proposition 4. The bundle $(\mathcal{E}_G^I, p_G^I, \mathcal{S}_G^I)$ is locally T -trivial.

Proof. The points of the space $\mathcal{E}_G^I$ may be identified with those embeddings $T \rightarrow T^I$ whose composition with the projection $\rho_i : T^I \rightarrow T$, given by the formula $\rho_i(\{z_i\}) = z_i$ for any $i \in I$, belongs to the semigroup A . The points of the space $(\mathcal{E}_G^I)_T$ may be identified with pairs (φ, z) , where φ is such an

embedding and $z \in \varphi(T)$; moreover $(\varphi', z) = (\varphi'', z)$ if $\varphi' = \varphi''g$, $g \in G$. The projection $(\mathcal{E}_G^I)_T \rightarrow \mathcal{S}_G^I$ consists in discarding the second element of the pair. Denote by

$$\pi_i : (\mathcal{E}_G^I)_T \rightarrow T \quad (i \in I)$$

the mapping given by the formula $\pi_i(\varphi, z) = \rho_i(z)$. The requirements in the definition of local T -triviality will be satisfied by the mapping $\pi_x : (\mathcal{E}_G^I)_T \rightarrow T$, which is defined for each point $x \in \mathcal{S}_G^I$ as the mapping coinciding with the mapping π_i , where $i \in I$ is such an element that there exists a point $y = \{a_i\} \in \mathcal{E}_G^I$ for which $p(y) = x$ and the element a_i is invertible in A .

Theorem 1. For every locally T -trivial bundle $\xi = (E, p, X)$ such that the cardinality of a basis of open sets of the space X does not exceed the cardinality of the set I , there exists a continuous mapping $\varphi : X \rightarrow \mathcal{S}_G^I$ such that the bundle ξ is equivalent to the bundle induced from $(\mathcal{E}_G^I, p, \mathcal{S}_G^I)$ by means of φ .

Proof. It is enough to construct a continuous mapping of the space E into the space $\mathcal{E}_G^I$, equivariant with respect to the action of the group G . For each point $y \in E$ there is defined a mapping $\eta_y : T \rightarrow E_T$ such that for any $x \in X$ with $p(y) = x$, the mapping $\pi_x \eta_y : T \rightarrow T$ is an invertible element of the semigroup A . From item 4° of the definition of an elementary G -object it follows that the elements $\pi_x \eta_y$ will be invertible in the semigroup A also for those $y \in E$ for which $p(y) \in U_x$, where U_x is some neighborhood of x . The neighborhoods U_x form coverings of the space, into which, by assumption, one may inscribe a covering whose cardinality does not exceed the cardinality of the set I . Therefore from all the mappings π_x one may select a collection of mappings $\{\pi_i\}$, indexed by the elements of the set i , such that for each point $y \in E$ at least one of the mappings $\pi_i \eta_y$ will be invertible in the semigroup A . It remains to define the mapping $\Phi : E \rightarrow \mathcal{E}_G^I$ by putting $\Phi(y) = \{\pi_i \eta_y\}$.

Thus, the bundle $\xi_G^I = (\mathcal{E}_G^I, p, \mathcal{S}_G^I)$ is universal for locally T -trivial principal G -bundles whose bases have a basis of neighborhoods whose cardinality does not exceed the cardinality of the set I .

Let us dwell in more detail on two basic examples of elementary G -objects. If G is a Lie group, $T = \mathbb{R}^n$, and A is the semigroup of all linear mappings, then $\mathcal{E}_G^I$ is the space of all linear embeddings

$$\mathbb{R}^n \rightarrow (\mathbb{R}^n)^I$$

such that the composition with one of the projections

$$(\mathbb{R}^n)^I \rightarrow \mathbb{R}^n$$

is nondegenerate. Recall that in the paper ⁽¹⁾, as the space $\mathcal{E}_G^I$ we considered the space of all linear embeddings

$$\mathbb{R}^n \rightarrow (\mathbb{R}^n)^I.$$

Thus the universal bundle obtained from the general construction set forth is a subbundle of the universal bundle from the paper ⁽¹⁾, and the base of the former is an everywhere dense open set in the base of the latter.

If $T = CG$, then the space $\mathcal{E}_G^I$ is obtained from $(CG)^I$ by deleting the single point $\{a_i\}$, where $a_i = *$ (the vertex of the cone) for all I . Let us note that if T is a set of n elements, then the space $\mathcal{E}_G^I$ is homeomorphic to the direct product of the $(n-1)$ -fold join

$$G * \dots * G$$

of the group G with the line $\mathbb{R}$, where the action of the group does not change the coordinate of the point on $\mathbb{R}$. Recall that Milnor's classical construction of the universal bundle for the group G consists in defining the action of the group on the space

$$E_n = \underbrace{G * \dots * G}_n,$$

and then as...

as the universal bundle for the group G one takes (E_G, p, B_G) , where $E_G = \varinjlim E_n$ and $B_G = E_G/G$. Thus, in the case $T = CG$ our construction gives an analogue of Milnor's universal bundle.

3. Definition. One says that a q -dimensional **characteristic class** of locally T -trivial principal G -bundles with coefficients in the topological G -module V is given if to each locally T -trivial principal G -bundle $\xi = (E, p, X)$ there is assigned an element $a(\xi) \in H^q(X; V)$, where V is the sheaf of germs of sections of the bundle with fiber V induced by the bundle ξ , and, moreover, for any map $\varphi : \xi' = (E', p', X') \rightarrow \xi'' = (E'', p'', X'')$ the equality $\bar{\varphi}^* a(\xi'') = a(\xi')$ holds. Here $\bar{\varphi} : X' \rightarrow X''$ is the map of bases corresponding to the map φ .

The characteristic classes form an abelian group, which is denoted by $H_{alg}^q(G; V)$.

Definition. The group $H_c^q(G; V)$ of continuous cohomology of Eilenberg-Mac Lane of the group G with coefficients in the topological G -module V is the q -th cohomology group of the complex

$$F_0 \xrightarrow{\partial_0} F_1 \xrightarrow{\partial_1} \dots,$$

where F_q is the group of continuous maps of the product $G \times \dots \times G$ (with q factors) into V , and

$$\partial_q f(g_1, \dots, g_{q+1}) = g_1 f(g_2, \dots, g_{q+1})$$

$$\sum_i (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{q+1}) + (-1)^{q+1} f(g_1, \dots, g_q).$$

Theorem 2. *For every paracompact group G the equality*

$$H_{alg}^q(G; V) = H_c^q(G; V)$$

holds. (In particular, the group $H_{alg}^q(G; V)$ does not depend on T .)

Proof of this theorem is based on the fact that if the characteristic classes a' and a'' coincide for all bundles ξ_G^I , then they are equal to one another. The following lemma plays the decisive role.

Lemma. *For every continuous real-valued function on the space $\mathcal{E}_G^I$ there is a subset $I' \subset I$ such that the difference $I \setminus I'$ is at most countable, and the restriction $f|_{\mathcal{E}_G^{I'}}$ is constant.*

The proof is the same as the proof of the analogous lemma in (1) (see Lemma 2.4).

Recall that in the paper (1) it was proved for closed subgroups G of the group $GL(n; \mathbf{R})$ that the group $H_{alg}^q(G; \mathbf{R})$ is isomorphic to the q -th cohomology group of the complex of G -invariant differential forms on the space $G/\hat{G}$, where $\hat{G} \subset G$ is a maximal compact subgroup. The isomorphism of this group with the group $H_c^q(G; V)$ was established by Hochschild and Mostow (5).

In conclusion we express our gratitude to Prof. A. Borel for his interest in our work and for pointing out the relevant literature.

Moscow State University
named after M. V. Lomonosov

Received
17 IV 1968

REFERENCES

1. I. M. Gelfand, D. B. Fuks, *Functional Analysis and Its Applications*, 1, No. 4 (1967).
2. I. M. Gelfand, D. B. Fuks, DAN, 176, No. 1 (1967).
3. I. M. Gelfand, D. B. Fuks, DAN, 177, No. 4 (1967).
4. W. T. Van Est, *Indagat. Math.*, 15, No. 5, 484 (1953).
5. G. Hochschild, G. D. Mostow, *Ill. J. Math.*, 6, No. 3, 367 (1962).
6. J. W. Milnor, *Ann. Math.*, 63, 272 (1956).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.