

DYNAMIC SYSTEMS WITH DELAY

V. K. Dubolar'

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Abstract

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MATHEMATICS

V. K. DUBOLYAR

DYNAMIC SYSTEMS WITH DELAY

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In the present note autonomous systems of differential equations with delayed argument are studied from the point of view of the general theory of dynamical systems ⁽¹⁾. This problem was posed in ⁽²⁻⁴⁾. Below, on the basis of the proposed axiomatization, the principal results in this direction are established, concerning invariant sets, minimal sets, and recurrent motions.

1°. In what follows, by the letter ρ we shall denote distance in metric spaces. Let R be a metric space; T the half-line $[0, +\infty)$; S a compact set contained in T ; Φ a metric space whose elements are continuous mappings of the compact set S into R , with the metric of uniform convergence on S ; f a continuous mapping $\Phi \times T$ into R . The ordered triple (Φ, T, f) will be called a **dynamical system with delay** if the following axioms are satisfied:

- 1) $f(\varphi, s) = \varphi(s)$ for all $\varphi \in \Phi$ and $s \in S$;
- 2) whatever $\varphi \in \Phi$ and $\tau \in T$ may be, the function φ^τ , defined by the relation $\varphi^\tau(s) = f(\varphi, \tau + s)$ for all $s \in S$, belongs to Φ ;
- 3) whatever $t \in T$, $\tau \in T$, and $\varphi \in \Phi$ may be, the equality

$$f(\varphi, t + \tau) = f(\varphi^\tau, t).$$

The proposed axiomatization is quite general. Properties 1)–3) are possessed by solutions of the differential equation with delayed argument

$$x'(t) = g(x(t), x(t - \tau)), \quad \text{where } 0 < \tau = \text{const},$$

provided that the right-hand side of the equation ensures existence, uniqueness, continuous dependence on the initial function, and continuability of the solution on $[\tau, +\infty)$. Properties 1)–3) are also possessed by solutions of the ordinary differential equation

$$x^{(n)} + p_1(t)x^{(n-1)} + \dots + p_{n-1}(t)x' + p_n(t)x = 0,$$

satisfying the boundary conditions $x(a_k) = A_k$, $a_k \geq 0$, $k = 1, 2, \dots, n$, provided the conditions of existence, uniqueness, continuous dependence on the boundary conditions, and continuability of solutions for all $t \geq 0$ are fulfilled.

Finally, let us give an example of a dynamical system with delay defined in a space of continuous functions. Let Ψ be some set of continuous functions defined on the half-line T with values in the metric space R ; S a compact set contained in T ; Φ the restrictions to S of functions of the set Ψ . We shall regard the space Φ as metric with the metric of uniform convergence on S . Suppose that Ψ satisfies the following conditions: a) if $\psi(t) \in \Psi$, then also $\psi(t + \tau) \in \Psi$ for any $\tau > 0$; b) if a sequence $\{\psi_n\}$ from Ψ converges uniformly on S , then it converges uniformly on every segment contained in T . Define the mapping $f : \Phi \times T \rightarrow R$ by putting $f(\varphi, t) = \tilde{\varphi}(t)$, where $\tilde{\varphi}$ is the extension to T of the function φ . We note that, by condition b), for each $\varphi \in \Phi$ the extension $\tilde{\varphi}$ is uniquely determined. Using the conditions

a) and b), it can be shown that the triple (Φ, T, f) constructed above is a dynamical system with delay.

2°. Let (Φ, T, f) be an arbitrary dynamical system with delay. Define the mapping $\sigma : \Phi \times T \rightarrow \Phi$ by setting $\sigma(\varphi, t) = \varphi^t$. It is easy to verify that the mapping σ is continuous, $\sigma(\varphi, 0) = \varphi$, and $\sigma[\sigma(\varphi, t_1), t_2] = \sigma(\varphi, t_1 + t_2)$ for any $\varphi \in \Phi$, $t_1 \geq 0$, $t_2 \geq 0$. Thus it becomes possible to pass from the given dynamical system with delay to the Birkhoff ⁽¹⁾ dynamical system $\sigma(\varphi, t)$, whose phase space is the set of initial functions Φ . Such a passage is useful in a number of cases. The idea of passing from solutions of differential equations with delayed argument to trajectories in the space of initial functions is contained in ⁽⁵⁾.

Theorem 1. *For any $\varepsilon > 0$, $l > 0$, and $\varphi \in \Phi$, there exists a $\delta > 0$ such that if $\psi \in \Phi$ and $\rho(\varphi, \psi) < \delta$, then $\rho(f(\varphi, t), f(\psi, t)) < \varepsilon$ for all $0 \leq t \leq l$.*

Theorem 1 is a generalization of the property of integral continuity known in the general theory of dynamical systems.

A set $F \subseteq \Phi$ is called invariant in Φ if, for any $\varphi \in F$ and $t \geq 0$, the relation $\varphi^t \in F$ holds. Using the system $\sigma(\varphi, t)$ defined above, one can show that the union (intersection) of any collection of sets invariant in Φ , as well as the closure of any set invariant in Φ , is an invariant set in Φ .

The function $f(\varphi, t)$, for fixed $\varphi \in \Phi$, will be called a motion, and the set of points $\{f(\varphi, t) : t \geq 0\}$ the trajectory of this motion. The trajectory of the motion $f(\varphi, t)$ will be denoted by the symbol $f(\varphi, T)$. Note that every finite arc of the trajectory $f(\varphi, T)$, as the continuous image of some segment from T , is compact.

A set $M \subseteq R$ is called invariant in R if there exists a set $\Psi \subseteq \Phi$ such that

$$M = \bigcup_{\psi \in \Psi} f(\psi, T),$$

i.e., if M consists of whole trajectories.

Theorem 2. *In order that a set $M \subseteq R$ be invariant in R , it is necessary and sufficient that there exist a set F , invariant in Φ , such that $M = f(F, 0)$.*

We outline the proof. If the set M is invariant in R , then

$$M = \bigcup f(\psi, T),$$

where Ψ is some set from Φ . The set

$$\{\psi^t : \psi \in \Psi, t \geq 0\}$$

is invariant in Φ . In this case $M = f(F, 0)$. Conversely, if $M = f(F, 0)$, where F is a set invariant in Φ , then

$$M = \bigcup_{\varphi \in F} f(\varphi, T),$$

i.e., M is invariant in R .

From the equality

$$\bigcup f(F_\alpha, t) = f\left(\bigcup F_\alpha, t\right),$$

valid for any $t \in T$ and $F_\alpha \subseteq \Phi$, and from Theorem 2, it follows that the union of any collection of sets invariant in R is an invariant set in R . However, there are examples showing that the intersection of invariant sets in R need not be an invariant set in R . One may also give an example of a set invariant in R whose closure no longer has this property. Nevertheless, the following assertion is valid: if $M \subseteq R$ and $F \subseteq \Phi$ is a set such that

$$\bigcup_{\varphi \in F} f(\varphi, T) \subseteq M,$$

then

$$\bigcup_{\varphi \in \overline{F}} f(\varphi, T) \subseteq \overline{M}.$$

In particular, if M is invariant in R , i.e. if

$$M = \bigcup_{\varphi \in F} f(\varphi, T),$$

where F is some set from Φ , then

$$\bigcup_{\varphi \in \overline{F}} f(\varphi, T) \subseteq \overline{M}.$$

What was noted above shows that, in their properties, dynamical systems with delay differ substantially from Birkhoff dynamical systems.

A dynamical system with delay (Φ, T, f) will be called **regular** if the following conditions are satisfied: 1) the compact S consists-

contains zero; 2) from every sequence of initial functions $\{\varphi_n\}$ from Φ , for which the corresponding sequence of points $\{\varphi_n(0)\}$ of the space R converges,

one can extract a subsequence converging in Φ . In this definition condition 2) is essential. It is satisfied, for example, if the space Φ is compact or when Φ consists of constant functions. In particular, the regularity conditions are satisfied if (Φ, T, f) is a Birkhoff dynamical system (i.e., Φ consists of constant functions, and the compact set S consists of a single point, coinciding with zero).

Theorem 3. *Suppose that the dynamical system (Φ, T, f) is regular. Then the closure of every invariant set in R is invariant.*

We give the scheme of the proof. If M is invariant in R , then on the basis of Theorem 2 the equality $M = f(F, 0)$ holds, where F is some invariant set in Φ . In this case $\bar{M} = f(\bar{F}, 0)$, whence the invariance of the closure $\bar{M}$ follows.

3°. A set $F \subseteq \Phi$ is called **minimal** in Φ if it is nonempty, closed, invariant, and contains no proper subset possessing these three properties.

Theorem 4. *In order that a nonempty invariant set $F \subseteq \Phi$ be minimal in Φ , it is necessary and sufficient that, for every function $\varphi \in F$, the closure in Φ of the set $\{\varphi^t : t \geq 0\}$ coincide with F .*

Theorem 5. *The following assertions hold: 1) every nonempty invariant closed compact set $F \subseteq \Phi$ contains some minimal set in Φ ; 2) if the space Φ is compact, then it contains some minimal set in Φ .*

The proof of Theorems 4 and 5 is carried out with the aid of the Birkhoff dynamical system $\sigma(\varphi, t)$ constructed in 2°. Its properties and its connection with the dynamical system with delay are used.

The motion $f(\varphi, t)$ is called **recurrent** if for every $\varepsilon > 0$ there exists $l > 0$ such that, whatever $t \geq 0$ may be, on every interval of length l there is a τ for which

$$\rho(f(\varphi, t + s), f(\varphi, t + s + \tau)) < \varepsilon$$

for all $s \in S$.

Theorem 6. *If φ belongs to a compact and minimal set in Φ , then the motion $f(\varphi, t)$ is recurrent.*

Theorem 7. *Suppose that the space R is complete. Then, if the motion $f(\varphi, t)$ is recurrent, the closure in the space Φ of the set $\{\varphi^t : t \geq 0\}$ is a compact and minimal set in Φ .*

The last two theorems are a generalization of the fundamental results of the general theory of dynamical systems known as Birkhoff's theorems (see ⁽¹⁾, pp. 402–404).

In the study of dynamical systems with delay, along with sets minimal in Φ , it makes sense also to consider sets minimal in R . In the case of Birkhoff systems these two notions coincide.

A set $M \subseteq R$ is called **minimal** in R if it is nonempty, closed, invariant, and contains no proper subset possessing the same properties.

Theorem 8. Suppose that the dynamical system with delay (Φ, T, f) is regular. Then, in order that a nonempty invariant set $M \subseteq R$ be minimal in R , it is necessary and sufficient that, for any trajectory $f(\varphi, T)$ contained in M , the equality

$$\overline{f(\varphi, T)} = M$$

hold.

The proof of this theorem is similar to the proof of the analogous assertion from ⁽¹⁾.

Theorem 9. Suppose that the dynamical system with delay (Φ, T, f) is regular, and that the set $M \subseteq R$ is compact and minimal

in R . Then there is a function $\varphi \in \Phi$ such that $\tilde{f}(\varphi, T) = M$, and the motion $f(\varphi, t)$ is recurrent.

The proof is based on Theorems 5, 6, and 8. We note that in Theorems 8 and 9 the regularity condition for the system $(\Phi, T, \tilde{f})$ is essential.

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Kishinev State
University

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