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Abstract

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THEORY OF ELASTICITY

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ASYMPTOTICS OF REISSNER'S EQUATIONS

IN THE NONLINEAR THEORY OF SYMMETRICALLY LOADED

SHELLS OF REVOLUTION

(Presented by Academician A. Yu. Ishlinskii on 31 I 1968)

1. E. Reissner ⁽¹⁾ obtained more exact equations for the finite symmetric deformation of thin shallow shells of revolution, abandoning assumptions concerning any smallness of the angles of rotation of a shell element as a result of deformation:

$$\begin{aligned}
 & (\Phi - \Phi_0)'' - (\Phi - \Phi_0)' \left(\frac{r_0}{\alpha_0} \right)' \bigg/ \frac{r_0}{\alpha_0} - \frac{\alpha_0^2 \cos \Phi}{r_0^2} (\sin \Phi - \sin \Phi_0) + \\
 & + \nu \frac{\alpha_0}{r_0} \Phi_0' (\cos \Phi - \cos \Phi_0) = \frac{\alpha_0^2}{r_0 D} (\Psi_H \sin \Phi - \Psi_V \cos \Phi), \\
 & \Psi_H'' + \Psi_H' \left(\frac{r_0}{\alpha_0} \right)' \bigg/ \frac{r_0}{\alpha_0} - \left[\frac{\alpha_0^2}{r_0^2} \cos^2 \Phi - \nu \frac{\alpha_0}{r_0} \Phi' \sin \Phi \right] \Psi_H = \\
 & = \frac{\alpha_0^2}{r_0} C (\cos \Phi - \cos \Phi_0) + \nu \frac{\alpha_0 \sin \Phi}{r} \Psi_H' + \\
 & + \left[\frac{\alpha_0^2}{r_0^2} \cos \Phi \sin \Phi + \frac{\alpha_0}{r_0} \nu \Phi' \cos \Phi \right] \Psi_V - \frac{\alpha_0}{r_0} (r_0^2 p_H)' - \nu \alpha_0^2 p_H \cos \Phi, \quad (1)
 \end{aligned}$$

$$\Psi_V = - \int r_0 \alpha_0 p_V d\xi, \quad C = Eh, \quad D = \frac{Eh^3}{12(1-\nu^2)}, \quad 0 < \nu < 0.5.$$

Here the surface of the shell is specified by the parametric equations $r_0 = r_0(\xi)$, $z_0 = z_0(\xi)$; $\Phi_0(\xi)$ is the angle which the shell element makes with the axis of abscissas before deformation at the corresponding point ξ , $\Phi(\xi)$ is the angle after deformation; Ψ_H and $\Psi_V \equiv (r_0 V)$ are, respectively, the horizontal and vertical components of stress; E is Young's modulus; ν is Poisson's ratio; $h = \text{const}$ is the shell thickness; p_H and p_V are, respectively, the horizontal and vertical components of the load, which depend on the load intensity $\rho(\xi)$ and the angle $\Phi(\xi)$.

Introduce the geometric parameter ε , characterizing the relative thinness of the shell: $\varepsilon = h/d\gamma$, where $\gamma^2 = 12(1-\nu^2)$ and d is the radius of the largest section perpendicular to the axis of revolution. We pass to dimensionless quantities. To this end we put

$$r_0 = rd, \quad z_0 = zd, \quad \alpha_0 = \alpha d,$$

$$\Psi = \Psi_H/d^2\gamma E\varepsilon^k, \quad p = p_H/E\gamma\varepsilon^k, \quad q = p_V/E\gamma\varepsilon^k, \quad (2)$$

$$T = \frac{\Psi_V}{d^2\gamma E\varepsilon^k} = - \int r\alpha q d\xi.$$

Substituting (2) into (1), we obtain Reissner's system of equations written in dimensionless form

$$\begin{aligned} \varepsilon^{3-k} \left\{ \frac{r}{\alpha} (\Phi - \Phi_0)'' + \left(\frac{r}{\alpha} \right)' (\Phi - \Phi_0)' - \frac{\alpha}{r} \cos \Phi (\sin \Phi - \sin \Phi_0) + \right. \\ \left. + \nu \Phi_0' (\cos \Phi - \cos \Phi_0) \right\} = \alpha (\Psi \sin \Phi - T \cos \Phi), \\ \varepsilon^{k-1} \left\{ \frac{r}{\alpha} \Psi'' + \left(\frac{r}{\alpha} \right)' \Psi' - \left[\frac{\alpha}{r} \cos^2 \Phi - \nu \Phi' \sin \Phi \right] \Psi \right\} = \\ = \alpha (\cos \Phi - \cos \Phi_0) + \varepsilon^{k-1} \nu \sin \Phi T' - \varepsilon^{k-1} (r^2 p)' + \varepsilon^{k-1} \left[\frac{\alpha}{r} \cos \Phi \sin \Phi + \nu \Phi' \cos \Phi \right] T - \nu \varepsilon^{k-1} r \alpha p \cos \Phi. \end{aligned} \quad (3)$$

For example, let us prescribe boundary conditions (reduced to dimensionless form) corresponding to a clamped shell along the edge:

$$\Phi = \Phi_0, \quad u \equiv d\varepsilon^{k-1} \left(\frac{r}{\alpha} \Psi' + r^2 p - \nu \Psi \cos \Phi - \nu T \sin \Phi \right) = 0 \quad \text{for } \xi = b, \quad (4)$$

and boundary conditions corresponding to an immovable hinged support:

$$M_\xi \equiv Eh^2\varepsilon \left[\frac{\Phi' - \Phi'_0}{\alpha} + \nu \frac{\sin \Phi - \sin \Phi_0}{r} \right] = 0; \quad u = 0 \quad \text{for } \xi = b. \quad (5)$$

2. For an approximate solution of equations (3), one can apply the Galerkin method in the form of P. F. Papkovich, analogously to how this is done within the framework of the semimembrane theory^(2,3). To this end, from the first equation (3) we find T and, substituting into the second equation, obtain:

$$\begin{aligned} \varepsilon^{k-1} \left\{ \frac{r}{\alpha} \Psi'' + \left(\frac{r}{\alpha} \right)' \Psi' - \frac{\alpha}{r} \Psi \right\} &= \alpha (\cos \Phi - \cos \Phi_0) + \varepsilon^{k-1} \nu \sin \Phi \cdot T' \\ -\varepsilon^{k-1} (r^2 p)' - \nu \varepsilon^{k-1} r \alpha p \cos \Phi + \frac{\varepsilon^2}{\alpha} \left[\frac{r}{\alpha} (\Phi - \Phi_0)'' + \left(\frac{r}{\alpha} \right)' (\Phi - \Phi_0)' \right. & \quad (6) \\ \left. - \frac{\alpha}{r} \cos \Phi (\sin \Phi - \sin \Phi_0) + \nu \Phi'_0 (\cos \Phi - \cos \Phi_0) \right] \left[\frac{\alpha}{r} \sin \Phi - \nu \Phi' \right] &\equiv f(\Phi). \end{aligned}$$

Let, further, $\{\varphi_k\}$ be an orthonormal basis in the corresponding Hilbert space. We shall seek the solution in the form

$$\Phi_n = \sum_{k=1}^n c_{nk} \varphi_k \quad (n = 1, 2, \dots). \quad (7)$$

Substituting (7) into the right-hand side of (6), from the resulting linear differential equation with the prescribed boundary conditions we find Ψ_n . The expression found for Ψ_n , as well as Φ_n , is then substituted into the first equation (3), and we require that the resulting expression be orthogonal to all φ_k ($k = 1, 2, \dots, n$); then we obtain a system for c_{nk} , for the solution of which one can apply the method⁽³⁾.

3. Equations (3) contain the natural small parameter ε multiplying the highest derivatives. This leads to the conclusion that asymptotic methods should be applied for the analysis of sufficiently thin shells. Moreover, the application of the Galerkin method in this case is difficult, since it requires the use of high approximations. The parameter ε may enter (3) in different ways depending on the manner of transition to dimensionless form. It seems natural to choose this method according to formulas (2), depending on the order of magnitude of the loads acting on the shell. Therefore, for loads of different orders of magnitude, different asymptotic expansions of the solutions will be constructed.

In what follows, for definiteness, a spherical shell under the action of hydrostatic pressure will be considered. For this, in (3) one must set $(0 \leq \xi \leq b < \pi)$

$$d = R, \quad r = \sin \xi, \quad z = -\cos \xi, \quad \alpha = 1, \quad p = -\rho \sin \Phi, \quad q = \rho \cos \Phi. \quad (8)$$

It should be noted, however, that everything that follows below is easily carried over to convex shells ($\Phi_0(\xi)$ is an increasing function, with $m\xi \leq \Phi_0(\xi) < \pi$, $m > 0$).

A. Let the intensity of the normal load ρ be less than or of order ε^2 . Then in (2), (3) one must take $k = 2$, i.e., in both equations (3), for the higher derivatives, there will be ε^1 . In this case the asymptotic expansions are constructed in powers of $(h/\gamma R)^{1/2}$.

Indeed, denote $\varepsilon = \mu^2$ and seek solutions in the form of series in powers of μ (the first iteration process $(^4, ^5)$)

$$\Phi = \sum_{s=0}^N \mu^s u_s, \quad \Psi = \sum_{s=0}^N \mu^s v_s. \quad (9)$$

Substituting (9) into (3) for $k = 2$, $\varepsilon = \mu^2$, and equating to zero the coefficients of $\mu^0, \mu^1, \mu^2, \dots$, we obtain systems of equations for determining v_s, u_s . Thus, to determine v_0, u_0 we have

$$v_0 \sin u_0 + \rho \cos u_0 \int_0^\xi \sin \xi \cos u_0 d\xi = 0, \quad \cos u_0 = \cos \xi. \quad (10)$$

Equations (10) have two solutions

$$1) \quad u_0 = \xi, \quad v_0 = -\frac{1}{4}\rho \sin 2\xi; \quad 2) \quad u_0 = -\xi, \quad v_0 = \frac{1}{4}\rho \sin 2\xi. \quad (11)$$

The first of them describes a shell shape close to the initial one and corresponding to a momentless stressed state; the second is close to the mirror-reflected shape and corresponds to the equilibrium state of the shell in the postcritical stage. Accordingly, the asymptotics is constructed in a neighborhood of each solution separately. The solutions (11), as well as v_s, u_s ($s \geq 1$), do not satisfy the boundary conditions (4) or (5). To satisfy them, the second iteration process $(^4, ^5)$ is used, for which the solutions are sought in the form

$$\Phi = \sum_{s=0}^N \mu^s (u_s + g_s), \quad \Psi = \sum_{s=0}^N \mu^s (v_s + h_s). \quad (12)$$

We substitute (12) into (3), use the equation for v_s, u_s , expand the known functions in Taylor series in a neighborhood of the point $\xi = b$, make the substitution $\xi = b - \mu t$, and equate to zero the coefficients of $\mu^0, \mu^1, \mu^2, \dots$. To determine g_0, h_0 we obtain the system of equations

$$\begin{aligned} \sin b g_0'' - \frac{v_0(b)}{\cos u_0(b)} \sin g_0 - h_0 \sin[u_0(b) + g_0] &= 0, \\ \sin b h_0'' - \cos[u_0(b) + g_0] + \cos u_0(b) &= 0 \end{aligned} \quad (13)$$

with boundary conditions (in the case of problem (3), (4)) for the first solution

$$g_0(0) = 0, \quad h_0'(0) = 0, \quad g_0(\infty) = h_0(\infty) = 0 \quad (14)$$

and for the second solution

$$g_0(0) = -2b, \quad h_0'(0) = 0, \quad g_0(\infty) = h_0(\infty) = 0. \quad (15)$$

We note that, for an approximate solution of problem (13), (15), the method ⁽⁶⁾ (pp. 34-38) works well. The equations for g_1 and h_i are not written out here because of their cumbersomeness. In the case of problem (3), (5) we obtain $h_0'(0) = g_0'(0) = 0$, and, consequently, from (13) we have $g_0 = h_0 = 0$. (An analogous result is obtained for the first solution of problem (3), (4).) Then for g_1, h_1 we obtain

$$g_1'' - \frac{1}{2}\rho g_1 - h_1 = 0, \quad h_1'' + g_1 = 0. \quad (16)$$

with boundary conditions for the first solution

$$g_1'(0) = 0, \quad h_0'(0) = -\frac{1}{2\rho}(1 - \nu), \quad g_1(\infty) = h_1(\infty) = 0 \quad (17)$$

and for the second solution in (11)

$$g_1'(0) = -2(1 + \nu), \quad h_1'(0) = \frac{1}{2\rho}(1 - \nu), \quad g_1(\infty) = h_1(\infty) = 0. \quad (18)$$

B. $\rho \sim O(\varepsilon)$, $k = 1$, and the small parameter ε^2 occurs only with derivatives Φ . Here the asymptotic expansions are constructed in powers of $h/R\gamma$. To determine v_0, u_0 we obtain the membrane equations:

$$\sin \xi v_0'' + \cos \xi v_0' + v_0 / \sin \xi = \cos u_0 - \cos \xi - \rho(\sin^2 \xi \sin u_0)', \quad (19)$$

$$v_0 \sin u_0 + \rho \int_0^\xi \sin \xi \cos u_0 d\xi \cos u_0 = 0.$$

For solution (18) one can apply a method combining S. A. Chaplygin's method and the method of power series ⁽⁵⁾. Let us write the first equations for the boundary layer:

$$h_0'' = 0, \quad g_0'' - k_1 \sin g_0 = 0,$$

$$k_1 = v_0(b) / \cos u_0(b) \sin(b) > 0, \quad (20)$$

$$h_0(\infty) = 0, \quad g_0(\infty) = 0,$$

whence $h_0 \equiv 0$, $g_0 = -4 \arctan C \exp[-\sqrt{k}(b - \xi)/\varepsilon]$, where the constant C is determined from the boundary condition for g_0 at $t = 0$.

C. $\rho \sim O(1/\varepsilon)$, $k = -1$. Here the asymptotic expansions are constructed in powers of $(h/R\gamma)^2$. The membrane equations have the form (18), but in the first equation the difference $\cos u_0 - \cos \xi$ is absent. To determine g_0, h_0 we obtain (20).

4. In the case of a hinged immovable fastening of the shell along the contour, the existence of solutions is proved for which asymptotic expansions are valid, and error estimates are given. Along the way, in case A the nonuniqueness of the number of solutions in the problem of equilibrium of a loaded spherical shell is established. To prove these facts, methods developed in works ^(5, 7) are used.

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Note: Figure translations are in progress. See original paper for figures.

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