

ON THE BEHAVIOR OF SOLUTIONS OF ROUGH PERIODIC AND AUTONOMOUS SYSTEMS

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Abstract

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MATHEMATICS

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ON THE BEHAVIOR OF SOLUTIONS OF ROUGH PERIODIC AND AUTONOMOUS SYSTEMS

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1. In the present note we consider certain questions concerning the behavior of solutions of systems of differential equations of the form

$$dx/dt = X(x, t), \quad (1,1)$$

where x and X are two-dimensional vectors. The vector function $X(x, t)$ is continuous and continuously differentiable with respect to x for all x and t ; moreover, it has period ω in t :

$$X(x, t + \omega) = X(x, t). \quad (1,2)$$

Denote by $\varphi(p, t_0, t)$ the solution of system (1,2) with initial data $t = t_0$, $x = p$, and by T the Poincaré transformation which assigns to the point p the point $\varphi(p, 0, \omega)$.

2. Let the point p be fixed under the transformation T^k . Linearizing system (1,1) in a neighborhood of the solution $x = \varphi(p, 0, t)$, we then obtain the system

$$dy/dt = P(t)y, \quad (2,1)$$

where the matrix $P(t)$ has period $k\omega$. Denote by ρ_1 and ρ_2 the roots of the characteristic equation of system (2,1). The fixed point p is called simple if $|\rho_j| \neq 1$ ($j = 1, 2$). In this case, if $|\rho_j| < 1$ ($j = 1, 2$), then p is a stable fixed point; if $|\rho_j| > 1$, then p is a completely unstable singular point.

If $0 < \rho_1 < 1 < \rho_2$, then p is a saddle, and if $0 > \rho_1 > -1 > \rho_2$, then p is a reverse saddle. Let p be a fixed point of the transformation T^k which is a reverse saddle; then, if it is regarded as a fixed point of the transformation T^{2k} , it turns out to be a saddle.

Let p be a saddle fixed point of the transformation T^k . From Perron's results ⁽³⁾ (see also ^(4, 5) in this connection) it follows that there exists a smooth arc $\Lambda^{(+)}$, containing p in its interior, such that

$$T^k \Lambda^{(+)} \subset \Lambda^{(+)}, \quad (2,2)$$

and such a smooth arc $\Lambda^{(-)}$, containing p in its interior, that

$$T^{-k} \Lambda^{(-)} \subset \Lambda^{(-)}. \quad (2,3)$$

The point p divides the arc $\Lambda^{(+)}$ into the parts $\Lambda_1^{(+)}$ and $\Lambda_2^{(+)}$.

Suppose that, for every natural number m , the transformation $T^{-km} \Lambda^{(+)}$ is defined. We shall call the curves

$$L_1^{(+)} = \bigcup_{m=1}^{\infty} T^{-km} \Lambda_1^{(+)}, \quad L_2^{(+)} = \bigcup_{m=1}^{\infty} T^{-km} \Lambda_2^{(+)}. \quad (2,4)$$

the stable separatrices of the saddle p . Analogously, we shall call the curves

$$L_1^{(-)} = \bigcup_{m=1}^{\infty} T^{km} \Lambda_1^{(-)}, \quad L_2^{(-)} = \bigcup_{m=1}^{\infty} T^{km} \Lambda_2^{(-)}. \quad (2,5)$$

the unstable separatrices.

Thus, four separatrices correspond to each saddle. Let p be a saddle fixed point of the transformation T^k , and let L be any one of its separatrices. We introduce on it a parameter s , varying on the half-axis $0 \leq s < +\infty$, so that $L(0) = p$.

The separatrices of saddles lying in sets invariant with respect to T possess special properties; we indicate some of them.

Theorem 2.1. Suppose that the set M of points of the plane has the following properties:

- (α) M is closed, bounded, and has zero Lebesgue measure.
- (β) The complement of M in the whole plane has only a finite number of components.
- (γ) $TM = M$.

Suppose that the stable (unstable) separatrix $L(s)$ is entirely contained in M . Suppose, moreover, that the separatrix L has, as $s \rightarrow +\infty$, a limiting point q , which also lies on a stable (unstable) separatrix L' , entirely contained in M .

Then every point of L' is a limiting point for L as $s \rightarrow +\infty$.

Theorem 2.2. Suppose that the set M has the properties (α) , (β) , (γ) of Theorem 2.1. Suppose, moreover, that the set M consists only of a finite number of points fixed with respect to certain powers of the transformation T , and of a finite number of stable (or unstable) separatrices.

Then, if the separatrix $L(s)$ lies in M , not one of its points can be a limiting point for it itself as $s \rightarrow +\infty$.

Theorem 2.3. Suppose that the set M satisfies the conditions of the preceding theorem.

Then the set M contains at least one separatrix which is not a limiting separatrix for any separatrix from M .

Theorem 2.4. Suppose that the set M satisfies all the conditions of Theorem 2.2; then the set M cannot contain an indecomposable continuum.

Theorem 2.5. Suppose that H is a plane region invariant with respect to T , and M is its boundary. Suppose further that the set M satisfies the conditions of Theorem 2.2.

Then, if the point p lies in M on the separatrix L and is attainable from H , then every point of L is attainable from H .

3. Let us now consider a dissipative (for the definition of dissipativity see $(^{2}, ^{5})$) system of the form (1.1). We introduce, as usual, the set I

$$I = \bigcap_{m=1}^{\infty} T^m R, \quad (3.1)$$

where R is a disk of sufficiently large radius with center at the origin.

Suppose that system (1.1) has a completely unstable fixed point O with respect to T . It is clear that $O \in I$. Let H be the region of negative stability of the point O . Put $J_1 = I \setminus H$. Suppose that the set J_1 has zero Lebesgue measure. Denote by G the complement of I in the whole plane, and by J the set of those points every neighborhood of which contains points of G and H . It is clear that $J \subset J_1$.

System (1.1), possessing the properties listed, was studied in detail in $(^{7}, ^{8})$. It was shown that the set J may have a very complicated structure (for example, it may be representable as a finite number of its elementary continua). We shall make one further assumption concerning the system assigned to us. In all that follows we shall assume that this system is rough and has only a finite number of periodic solutions.

Under the assumptions stated, the following assertions concerning the structure of the set J and the behavior of the solutions beginning on it are valid.

Theorem 3.1. The sets J_1 and J consist only of a finite number of stable and saddle points and a finite number of unstable separatrices.

This assertion, together with Theorems 2.1–2.5, makes it possible to establish the following properties of the system under consideration.

Theorem 3.2. The set J cannot be represented as a finite number of its elementary continua.

Theorem 3.3. The inner and outer rotation numbers on the set J coincide with one another and are rational.

Theorem 3.4. If the rotation numbers on the set J have the form m/k , then there exists a point $p \in J$ such that $T^k p = p$ and which is attainable both from G and from H .

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Note: Figure translations are in progress. See original paper for figures.

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