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Abstract

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MATHEMATICS

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BOUNDARY-VALUE PROBLEMS OF S. L. SOBOLEV TYPE FOR ELLIPTIC OPERATORS

(Presented by Academician I. G. Petrovskii on 29 III 1968)

1. Let X be a manifold (closed, for simplicity) and let $Y \subset X$ be its submanifold of codimension $\nu \geq 1$.

By problems of Sobolev type, or, briefly, S-problems, we shall mean the problem of finding a function satisfying an elliptic equation on $X \setminus Y$ and assuming on Y certain boundary values.

As far as I know, problems of this type were first considered by S. L. Sobolev ⁽³⁾, who proved unique solvability of the Dirichlet problem for the polyharmonic equation $\Delta^m u = 0$. In the paper ⁽⁴⁾ a general theory of such problems was constructed for the case of closed submanifolds of an arbitrary (elliptic) pseudodifferential operator on the manifold X and arbitrary pseudodifferential operators on the submanifold Y . More precisely, in that paper spaces and conditions on the coefficients of the equation were indicated that are necessary and sufficient in order that the S-problem in these spaces be normally solvable, i.e., uniquely solvable up to finite-dimensional spaces.

In the present paper an analogous question is investigated for the case when the submanifold Y of the S-problem has a boundary.

If we realize pseudodifferential expressions as an unbounded operator acting in spaces in which the corresponding S-problem is normally solvable, then such an operator will, obviously, be Fredholm, i.e., will have finite-dimensional kernel and cokernel. Thus, the theorem on normal solvability of such a problem may be regarded as a finiteness theorem for the corresponding operator. The topological aspects of boundary S-problems, and in particular the computation of the index of the corresponding (Fredholm) operator, I shall consider elsewhere.

2. Thus, let X be (for simplicity) a closed manifold of dimension N , and let $Y \subset X$ be its submanifold of codimension ν :

$$1 \leq \nu \leq \dim X - 1.$$

The cases $\nu = \dim X$, $\nu = \dim X - 1$ are atypical and differ somewhat from the general case. For example, if $\nu = \dim X$, i.e., the manifold Y degenerates to a point, then no boundary values can be prescribed on it, and nevertheless the problem will be normally solvable. If $\nu = \dim X - 1$, then the boundary of the manifold Y degenerates to a point, and nothing can be prescribed on it. Next, we note that for $\nu > 1$, normally solvable problems are only S-problems (at least in Sobolev spaces), whereas in the case $\nu = 1$, besides problems of type S, one may also consider other problems (of conjugation-problem type).

3. In the paper ⁽⁵⁾ we introduced the notions of boundary and coboundary operators. Let us recall them.

For any finite-dimensional complex vector bundle $E \rightarrow X$ over a smooth (possibly with boundary) manifold X and any real number $s \in \mathbb{R}^1$, we shall denote by $\Gamma^s(X, E)$ the Sobolev space ⁽²⁾ of sections of the bundle E .

If $i : Y \subset X$ is the embedding of a submanifold Y of codimension ν in a manifold X , then for $s > \nu/2$ there is a continuous mapping

$$\delta = \delta_Y : \Gamma^s(X, E) \rightarrow \Gamma^{s-\nu/2}(Y, i^*E),$$

which assigns to each section $f \in \Gamma^s(X, E)$ its restriction $f|_Y$ to the manifold Y . We shall call such an operator an **elementary boundary operator**. A **general boundary operator** is a composition

$$\Gamma^s(X, E_1) \xrightarrow{D} \Gamma^t(X, E_2) \xrightarrow{\delta} \Gamma^{t-\nu/2}(Y, i^*E_2)$$

of some pseudodifferential operator D and the restriction operator. It is easy to see that a boundary operator is continuous if its components are continuous.

The operator adjoint (in the sense of the L_2 -scalar product) to an elementary boundary operator is

$$\delta^* = \delta_Y^* : \Gamma^{-s+\nu/2}(Y, i^*E) \rightarrow \Gamma^{-s}(X, E),$$

which we shall call an **elementary coboundary operator**. A **general coboundary operator** is a composition

$$\Gamma^{-s+\nu/2}(Y, i^*E_1) \xrightarrow{\delta^*} \Gamma^{-s}(X, E_1) \xrightarrow{D} \Gamma^{-t}(X, E_2)$$

of an elementary coboundary operator and some pseudodifferential operator D .

Let us now consider the problem

$$D_X u = f_X, \quad x \notin Y \cup \partial Y \quad (1)$$

of finding a section $u \in \Gamma^s(X, E_1)$ satisfying, off the manifold Y , equation (1) for $f_X \in \Gamma^{s-m}(X, E_2)$, where m is the order of the expression D_X . First of all, note that if $s - m + \nu/2 \geq 0$, then equation (1) is equivalent to the equation on

the whole manifold X , and, thus, problem (1) is normally solvable if and only if the expression D_X is elliptic.

Now let $s - m + \nu/2 < 0$. Then, since in the space $\Gamma^{s-m}(X, E_2)$ there are nontrivial elements concentrated on the submanifold Y , and since $\dim Y > 0$, equation (1) has, generally speaking, an infinite-dimensional space of solutions. To remove this nonuniqueness one must additionally impose a certain number of boundary conditions. In order to understand the situation precisely, we proceed (following the principle of locality) as follows. We shall straighten the boundary near a typical point of the submanifold Y . Then we arrive at a problem in the space $\mathbf{R}^N$ and see that on the manifold Y one must impose a certain number of boundary operators connected with the operator D_X by a certain algebraic condition (see ⁽⁴⁾). In the case when the manifold Y had no edge, this condition, together with the ellipticity condition of the operator D_X , ensured normal solvability of the equation. In invariant terms, fulfillment of this condition means that normal solvability of our problem is equivalent to normal solvability of two problems: the problem for an elliptic operator on the manifold X and the problem for an elliptic operator on the manifold Y . In the case when the manifold Y has no boundary (recall that we have everywhere assumed that the manifold X is closed), these problems are normally solvable. If, however, the manifold Y has an edge ∂Y and $\dim \partial Y > 0$, this is, generally speaking, no longer the case. To understand the situation, we again consider a typical point on the (closed) manifold ∂Y . Again, in accordance with the principle of locality, we straighten the boundary near such a point. Let $t^1, \dots, t^\nu, x^1, \dots, x^{n-1}, x^n = (t, x', x_n)$ be coordinates in the space $\mathbf{R}^n$, with ...

the (open) manifold Y is given by the equation $t = 0, x^n > 0$. Freezing the coefficients of equation (1) and passing in it to the Fourier image with respect to the variables $x^1, \dots, x^{n-1} = x'$, we obtain an equation which we shall conventionally write in the form

$$D_X(t, \xi', x_n)u(t, \xi', x_n) = f. \quad (2)$$

This equality holds for all x^n and t , except for points lying on the half-line $t = 0, x^n > 0$. For the subsequent investigation the right-hand side of equation (2) is inessential, and we shall assume $f(t, \xi', x_n) \equiv 0$. Hence it follows that in the whole space $\mathbb{R}^N$ equation (2) can be written in the form

$$D_X(t, \xi', x^n)u(t, \xi', x_n) = Cv(\xi', x_n), \quad (3)$$

where the matrix C has the form

$$C = \left(\begin{array}{c|c|c} 1 & & \\ \hline & \ddots & \\ \hline & & \frac{\partial^j \delta(t)}{\partial t^j} \\ \hline & & \ddots \\ \hline & & \frac{\partial^j \delta(t)}{\partial t^j} \\ \hline \end{array} \right).$$

Here δ is the Dirac δ -function; $j = (j_1, \dots, j_\nu)$, $|j| = \sum j_k \leq l^\nu$ (see (5)), $v(\xi', t)$ is a (vector-)function concentrated on the half-axis $x^n > 0$ with as yet arbitrary components. Let now $\xi' \neq 0$. Then the solution of equation (3) has the form

$$u(t, \xi', x^n) = \int_{R^1} \int_{R^{n-1}} \exp\{-i[(t, \tau) + (x^n, \xi_n)]\} D_X^{-1}(\tau, \xi', \xi_n) C(\tau) v(\xi', \xi_n) d\tau d\xi_n.$$

Applying to the left- and right-hand sides of the written equality the boundary operators $\delta_Y D_Y(t, \xi', x^n)$, we obtain, for any $\xi' \neq 0$,

$$g = \frac{1}{2\pi} \int_{R^1} \exp[-i(x^n, \xi_n)] \int_{R^{n-1}} D_Y(\tau, \xi', \xi_n) D_X^{-1}(\tau, \xi', \xi_n) C(\tau) d\tau v(\xi', \xi_n) d\xi_n,$$

where $g = \delta_Y D_Y u$. Thus we arrive at a system of pseudodifferential equations

$$g(x^n) = \frac{1}{2\pi} \int_{R^1} \exp[-i(x^n, \xi_n)] \sigma(\xi', \xi_n) v(\xi', \xi_n) d\xi_n \quad (4)$$

with symbol

$$\sigma(\xi', \xi_n) = \int_{R^{n-1}} D_Y(\tau, \xi', \xi_n) D_X^{-1}(\tau, \xi', \xi_n) C(\tau) d\tau.$$

The unique solvability of equation (4) is what interests us.

Let us first observe that, since the solutions $v(\xi', \xi_n)$ of this equation must be concentrated for $x^n \geq 0$, the equation obtained is essentially an equation on a half-line.

From what was said above it follows that the symbol of this operator is not equal to zero for any ξ_n . (Recall that $\xi' \neq 0$.) The theory of such equations in Sobolev spaces was developed in a series of works by M. I. Vishik and G. I. Eskin ⁽¹⁾. This theory is based on the principle of factorization and consists in the following. It is known that the matrix $\sigma(\xi', \xi_n)$ admits the factorization

$$\sigma(\xi', \xi_n) = \sigma_+(\xi', \xi_n) \sigma_-(\xi', \xi_n)$$

with factors nondegenerate for $\xi' \neq 0$ and analytic, respectively, in the upper and lower complex half-planes. It is shown—

It follows from (1) that the operator Σ_+ with symbol $\sigma_+(\xi', \xi_n)$ is Fredholm. Let $\dim \text{Ker } \Sigma_+$ and $\dim \text{Coker } \Sigma_+$ be the dimensions of its kernel and cokernel. Then, specifying $\dim \text{Ker } \Sigma_+$ boundary and $\dim \text{Coker } \Sigma_+$ coboundary conditions at the point $x_n = 0$, connected by a natural algebraic condition of the Shapiro-Lopatinskii type, we obtain unique solvability of equation (4).

4. In the preceding paragraph we have completely defined the boundary-value problem that we shall call a boundary-value problem of type S , or, briefly, an S -problem. (In order not to burden the exposition, I do not write out the spaces of the right-hand sides of the S -problem. The reader will have no difficulty filling this gap.)

We now formulate the main result of the paper.

Theorem. *Suppose that the algebraic conditions on the coefficients of the problem formulated in § 3 are satisfied. Then the boundary S -problem is normally solvable.*

5. Lack of space does not allow us to consider constructions that generalize S -problems and are of great importance in considering topological aspects of the theory, namely elliptic morphisms ⁽⁷⁾ for manifolds with boundary. These questions will be considered in a subsequent paper.

I discussed the questions touched upon in this work with V. V. Grushin; I express my gratitude to him for his attention.

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