

ON THE EMBEDDING OF RINGS IN FIELDS

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Abstract

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MATHEMATICS

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ON THE EMBEDDING OF RINGS IN FIELDS

(Presented by Academician A. I. Mal' tsev on 12 IX 1966)

I. In this note a proof is given of the following result:

Theorem. *There exists an (associative) ring without zero divisors, not embeddable in a field, whose multiplicative semigroup of nonzero elements is embeddable in a group.*

This is a solution of a problem posed by A. I. Mal' tsev.

II. Following (2), let us call a ring K an SN -ring if K is an algebra with 1 over a field F , given in some system of generators Ξ by relations of the form

$$w_i h_i = u_i f_i, \quad (1)$$

where $i \in I$, $w_i, h_i, u_i, f_i \in \Xi$, satisfying the conditions:

S1) $\{w_i, u_i\} \cap \{h_j, f_j\} = \emptyset$.

S2) The words $w_i h_i, u_i f_i, i \in I$, are pairwise distinct.

N) If in K there is an equality $AB = CD$, where A, B, C, D are nonzero linear combinations of generators from Ξ , then either $A = CX, D = XB, X \in K$, or there is an equality $xy = zt$ among (1) such that the relations hold:

$$A = aX(x + \alpha z), \quad B = (y + \beta t)Y, \quad C = X(z + \beta x), \quad D = (t + \alpha y)Ya, \quad (2)$$

where $0 \neq X, Y, a \in F, \alpha, \beta \in F$.

In (1) it is proved that, as a basis of the algebra K , one may take the element 1 and all words in the generators not containing the left-hand sides of the relations (1) (assuming that in (1) the left- and right-hand sides are fixed). In (2) it is proved that SN -rings have no zero divisors.

By $\overline{SN}$ we shall denote the totality of completions $\overline{K}$ of rings $K \in SN$ with respect to the natural norm. The ring $\overline{K}$ is the ring of formal infinite series of the form

$$A = \sum_{i=0}^{\infty} A_i,$$

where A_i are homogeneous elements of degree i .

III. Let $\overline{K}$ be an arbitrary $\overline{SN}$ -ring. Introduce the following sets of elements of the ring $\overline{K}$: Γ is the set of all invertible elements of the ring $\overline{K}$; P is the set of elements of the form

$$p(w, u, A) = w + uA, \quad p(u, w, C) = u + wC, \quad (3)$$

where $wh = uf$ is an arbitrary one of the relations (1); A is an arbitrary element of $\overline{K}$ such that all basic words in A do not begin with f and do not end with w ; all basic words in C do not begin with h and do not end with w ; Q is the set of elements of the form

$$q(f, h, A) = f + Ah, \quad q(h, f, C) = h + Cf, \quad (4)$$

where $wh = uf$ is an arbitrary one of the relations (1), and A and C are the same as above; R is the set of all irreducible elements (one from each class of associated elements), each of which is not associated with any element from $P \cup Q$.

Let us single out in the sets P, Q the subsets $P \subseteq P_1$, P^* , $Q \supseteq Q_1$, Q^* . Namely, into P_1 and Q_1 we place elements of the form (3) and (4) for which A and C are not invertible; into P^*, Q^* we place elements of the form $P^* = \{p(w, u, A^*)\}$, $Q^* =$

$$= \{q(f, h, A^*)\},$$

where A^* are invertible elements. For a fixed relation

$$wh = uf \quad (5)$$

we shall regard the sets of all noninvertible elements A and C occurring in (3) and (4), as well as the set of all invertible elements A^* from (3) and (4), as completely ordered:

$$A_1, A_2, \dots; C_1, C_2, \dots; A_1^*, A_2^*, \dots.$$

After this we introduce new notation for the elements of the sets P_1, Q_1, P^*, Q^* , whose meaning will be clear from the examples:

$$p(w, u, A_i) = p_i(w, u) = p_{iw}, \quad q(f, h, A_i^*) = q_i^*(f, h) = q_{if}^*.$$

We define the group Γ by a system of generators consisting of all elements of this group and by some system of relations $X_i = 1$. Denote by $\overline{K}^*$ the multiplicative semigroup of nonzero elements of the ring $\overline{K}$.

Theorem 1 ⁽³⁾. *The semigroup $\overline{K}^*$ is given in the system of generators*

$$P_1 \cup Q_1 \cup P^* \cup Q^* \cup R \cup \Gamma \quad (6)$$

by the following system of defining relations, constructed for all relations (5) among (1) and for all indices k, i, n, m :

$$1. \quad p_k(w, u)X_{ki}q_i(h, f) = p_i(u, w)Y_{ik}q_k(f, h). \quad (7)$$

$$2. \quad p_k^*(w, u)X_{ni}^*q_i(h, f) = p_i(u, w)Y_{in}^*q_n^*(f, h). \quad (8)$$

$$3. \quad p_k(w, u)\overline{X}_{kn}q_n^*(f, h) = p_n^*(w, u)\overline{Y}_{nk}q_k(f, h). \quad (9)$$

$$4. \quad p_n^*(w, u)X_{nm}^{**}q_m^*(f, h) = p_m^*(w, u)Y_{mn}^{**}q_n^*(f, h), \quad (10)$$

where $A_n^* - A_m^*$ is an invertible element,

$$X_{nm}^{**} = Y_{mn}^{**} = (A_n^* - A_m^*).$$

$$5. \quad \alpha(a + L\alpha) = (a + \alpha L)\alpha, \quad (11)$$

where $0 \neq a \in F$, $\alpha \in P_1 \cup P^* \cup Q_1 \cup Q^* \cup R$.

$$6. \quad X_i = 1. \quad (12)$$

Here we have used the notation:

$$X_{ki} = (1 - C_i A_k)^{-1}, \quad Y_{ik} = (1 - A_k C_i)^{-1}, \quad X_{ni}^* = (1 - C_i A_n^*)^{-1}, \quad Y_{in}^* = (1 - A_n^* C_i)^{-1},$$

$$\overline{X}_{nk} = \overline{Y}_{kn} = (A_n^* - A_k)^{-1}.$$

IV. We now recall the definition of the rings $F(Q_n)$ from ⁽¹⁾. The ring $F(Q_n)$, $n \geq 2$, is an algebra with 1 over the field F , given in the system of generators

$$\{a_i, b_i, c_i, s_i, t_i, 1 \leq i \leq n; v_0, v_1\}$$

by the defining relations

$$a_i s_i = c_i v_0, \quad b_i s_{i+1} = c_i v_1, \quad b_i t_{i+1} = a_i t_i,$$

where $1 \leq i \leq n$, $s_{n+1} = s_1$, $t_{n+1} = t_1$. In ⁽²⁾ it was proved that the rings $F(Q_n)$ are SN -rings. If F is a field of characteristic $p \neq 0$, $n = p^k$, then, as shown in ⁽¹⁾, the rings $F(Q_n)$ are not embeddable in fields. On the other hand, it will follow from what follows that the semigroups $GF(2)(Q_n)^*$ are embeddable in groups. Therefore the ring $GF(2)(Q_4)$ is the simplest example of a ring giving an answer to A. I. Mal'cev's problem.

V. Let again $\bar{K}$ be an arbitrary $\overline{SN}$ -ring, and let G be the group of fractions of the semigroup $\bar{K}^*$, i.e. the group given by the generators (6) and the relations (7)–(12). Fix some relation (5). In the group G the equalities

$$\begin{aligned} X_{n1}^* q_{1n} Q_{1nf}^* q_{1h}^{-1} X_{ni}^{*-1} P'_{1nw} \bar{Y}_{nk} q_{kf} Q_{2nf}^* q_{sf}^{-1} \bar{Y}_{ns}^{*-1} P'_{2nw}, \\ P'_{3nw} \bar{Y}_{nk} q_{kf} Q_{3nf}^* q_{sf}^{-1} \bar{Y}_{ns}^{*-1} P'_{4nw} X_{n1}^* q_{1h} Q_{4ng}^* q_{ih}^{-1} X_{ni}^{*-1}, \end{aligned} \quad (13)$$

hold, if the following relations hold:

$$\begin{aligned} 1. \quad P_{1nw}^* &= P_{iu} P_{kw}. \\ 2. \quad Y_{ln}^* Q_{1nf}^* Y_{in}^{*-1} P'_{iu} &= P'_{lu} Y_{lk} Q_{kf} Y_{ik}^{-1}. \\ 3. \quad P'_{kw} \bar{X}_{kn} Q_{2nt}^* \bar{X}_{sn}^{*-1} &= X_{ki} Q_{in} X_{si}^{-1} P'_{sw}. \\ 4. \quad Q'_{kf} Q'_{in} &= Q'_{ln} Q'_{sf}. \\ 5. \quad P_{1u} &= P_{3nw}^* P_{2kw}. \\ 6. \quad P_{1iu} &= P_{sw}^* P_{2nw}^*. \\ 7. \quad P'_{1kw} X_{kl} Q_{ln} X_{sl}^{-1} &= \bar{X}_{kn}^* Q_{3nf}^* \bar{X}_{sn}^{*-1} P'_{1sw}. \\ 8. \quad Y_{ls} Q_{sf} Y_{is}^{-1} P'_{1iu} &= P'_{1lu} Y_{ln}^* Q_{4nf}^* Y_{in}^{*-1}. \\ 9. \quad P_{1sw} P_{1lu} &= P_{4nw}^*. \end{aligned} \quad (14)$$

The notation introduced here will be clear from the following examples: $Q_{1nf}^* = 1 + q_{nf}^* L_1$, $Q_{1nf}' = 1 + L_1 q_{nf}^*$, $Q_{2nf}^* = 1 + q_{nf}^* L_2$, where $L_1, L_2 \in \bar{K}$, $P_{iu} = 1 + p_{iu} L$, $P_{iu}' = 1 + L p_{iu}$, $L \in \bar{K}$.

In G the equalities

$$\begin{aligned} \bar{X}_{sn}^{*-1} p_{sw}^{-1} \bar{P}_{1nw}^* p_{kw} \bar{X}_{kn}^* Q_{1nf}^* Y_{in}^* p_{iu}^{-1} P_{2nw}^* p_{lu} Y_{ln}^* Q_{2nf} = \\ = Q_{3nf}^* Y_{in}^{*-1} p_{iu}^{-1} P_{3nw}^* p_{lu} Y_{ln}^* Q_{4nf}^* \bar{X}_{sn}^{*-1} p_{sw}^{-1} P_{4nw}^* p_{kw} \bar{X}_{kn}^*, \end{aligned} \quad (15)$$

also hold, if relations analogous to those given in (14) hold.

VI. Let us give one more collection of equalities of the group G . Suppose that among the defining relations of the ring K under consideration there is a triple of equalities of the form:

$$wh = uf, \quad w_1 h_1 = uf_1, \quad wh_2 = w_1 f_2. \quad (16)$$

Fix the equalities (16) and introduce the notation:

$$\begin{aligned} \bar{p}_{nw} &= w + uf_1 L_n, & \bar{q}_{nf} &= f + f_1 L_n h, & \bar{q}_{nf_2} &= f_2 + h_1 L_n h_2, \\ \bar{p}_{iu} &= u + wh_2 M_i, & \bar{q}_{ih} &= h + h_2 M_i f, & \bar{q}_{ih_1} &= h_1 + f_2 M_i f_1, \\ \bar{p}_{kw_1} &= w_1 + uf S_k, & \bar{q}_{kf_1} &= f_1 + f S_k h, & \bar{q}_{kh_2} &= h_2 + h_1 S_k f_2, \end{aligned} \quad (17)$$

where L_n, M_i, S_k are arbitrary nonzero elements of the ring $\bar{K}$, not ending in w, u, w_1 , respectively. (We regard the sets $\{L_n\}$, $\{M_i\}$, $\{S_k\}$ as completely ordered.)

In the group G the equalities

$$\begin{aligned} P'_{1nw} X'_{nj} Q'^{-1}_{nf} \bar{q}^{-1}_{in} X'^{-1}_{ni} P'_{nw} X'''_{nl} q_{lh_2} Q'^{-1}_{nf_2} \bar{q}^{-1}_{kh_2} X'''^{-1}_{nk} = \\ = X'''_{nl} q_{lh_2} Q'^{-1}_{nf_2} \bar{q}^{-1}_{kh_2} X'''^{-1}_{nk} P'_{3nw} X'_{ni} \bar{q}_{jh} Q'^{-1}_{nf} \bar{q}^{-1}_{ih} X'^{-1}_{ni} P'_{2hw}, \end{aligned} \quad (18)$$

hold, if relations analogous to those given in (14) are satisfied.

The meaning of the notation introduced will be clear from examples:

$$P_{1nw} = 1 + \bar{p}_{nw} A_1, \quad P'_{1nw} = 1 + A_1 \bar{p}_{nw}, \quad A_1 \in \bar{K}, \quad X'_{ni} = (1 - h_2 M_i f L_n)^{-1},$$

$$Y'_{in} = (1 - f_1 L_n h_2 M_i)^{-1}, \quad X''_{ki} = (1 - f_2 M_i f S_k)^{-1}, \quad X'''_{nk} = (1 - h S_k h_1 L_n)^{-1}.$$

VII. Let now $F = GF(2)$. Fix some semigroup $\overline{F(Q_n)}^*$, $n \geq 2$, and let G be its group of quotients. We shall show how, on the basis of what has preceded, it is proved that the semigroup $\overline{F(Q_n)}^*$ is a subsemigroup of the group G . First of all, include among the defining relations of the group G the relations (13), (15), and (18). Next define a finite sequence of groups G_i , $1 \leq i \leq 8$. All these groups are obtained according to the rule: among the generators of the group G some subset is singled out and all defining relations from the elements of this subset are considered. Therefore, in order to define the groups G_i , it is enough to specify the sets of generators Σ_i of each of these groups. We have: $\Sigma_1 = \Xi \cup \Gamma$; Σ_2 is obtained from Σ_1 by adjoining all elements of the form $p_i(u, w)$, $q_i(h, f)$ such that they do not enter the sets (18), constructed for all relations (17) obtained from the relations of the ring $F(Q_n)$ by fixing the number i , $1 \leq i \leq n$; Σ_3 is obtained by adjoining to Σ_2 elements of the form $\bar{p}_n(u, w)$, $\bar{q}_n(h, f)$, $\bar{q}_n(h_1, f_1)$, constructed for all relations (17);

Σ_4 is obtained from Σ_3 by adding all elements $p_n(w, u)$, $q_n(f, h)$ that do not belong to the set (18); Σ_5 consists of the elements of Σ_4 and elements of the form $\bar{p}_n(w_1, u)$, $\bar{q}_n(f_1, h_1)$, $\bar{q}_n(h_2, f_2)$, constructed for all triples (17); Σ_6 is obtained by adding to Σ_5 the elements $\bar{p}_n(w, u)$, $\bar{q}_n(f, h)$, $\bar{q}_n(f_2, h_2)$, constructed for all triples (17); finally,

$$\Sigma_7 = \Sigma_6 \cup P^* \cup Q^*, \quad \Sigma_8 = \Sigma_7 \cup R.$$

In the groups G_i , $1 \leq i \leq 8$, a canonical form of elements is constructed successively, analogously to how this is done in the works ⁽¹⁾ and, especially, ⁽⁴⁾. As a result it turns out that every word of the group G is equal to one and only one word having canonical form. Moreover, if W is a positive word of the group G (a word in the alphabet (6)), then to obtain its canonical form it is enough to carry out on W a certain number of “semigroup” transformations (transformations corresponding to the equalities (7)–(12)). This proves the required result.

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