

LINEAR DIFFERENTIAL GAMES

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1967

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Abstract

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UDC 517.91

MATHEMATICS

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LINEAR DIFFERENTIAL GAMES

1. Let the motion of the vector z in an n -dimensional Euclidean space R be described by the linear vector differential equation

$$dz/dt = Cz + U(u) - V(v), \quad (1)$$

where u and v are control parameters whose values lie on the $(\nu-1)$ -dimensional unit sphere K ; C is a constant matrix; U and V are analytic mappings of the sphere K into the space R . Suppose that in R an $(n-\nu)$ -dimensional vector subspace M is given. We shall say that all these data describe a **linear differential game** (1) (see (1)).

We shall say that the game (1) can be completed on a certain set $A \subset R$ if, for any initial value $z_0 \in A$ of the vector z and for an arbitrary piecewise-continuous variation of the control parameter $v(t)$, one can choose such a variation of the control parameter $u(t)$ that the point z reaches the subspace M in a time not exceeding the number $T(z_0) \geq 0$; moreover, in order to find the value of the parameter $u(t)$ at each time instant t , only the values $z(t)$ and $v(t)$ at the same time instant t are used, and the values of z and v at times following t are not used.

In the present note we indicate some very simple conditions sufficient for the completion of a linear differential game.

2. Let L be the orthogonal complement in R to the subspace M . We shall denote by πz the orthogonal projection of any vector $z \in R$ onto the subspace L . The right-hand side of equation (1) determines the following two mappings of the sphere K into L :

$$y = \pi(e^{-\tau C}U(u)), \quad (2)$$

$$y = \pi(e^{-\tau C}V(v)). \quad (3)$$

Everywhere below we shall assume that τ is a negative number.

We shall suppose that the following condition is fulfilled.

Condition a). The images of the sphere K under the mappings (2) and (3) are convex, locally convex $(\nu - 1)$ -dimensional hypersurfaces in L . (Local convexity is understood here in the differential-geometric sense, as the definiteness of the second quadratic form.)

Consider the scalar products

$$\varphi \cdot \pi(e^{-\tau C} U(u)), \quad \varphi \cdot \pi(e^{-\tau C} V(v)), \quad (4)$$

where φ is an arbitrary unit vector from L , which we shall regard as a point of the sphere K , and the sphere K as the unit sphere in L . Let $u = u(\tau, \varphi)$, $v = v(\tau, \varphi)$ be the points giving the maximum of the corresponding expressions (4) for fixed τ, φ , with respect to u , respectively to v . Construct the mapping

$$y = \pi[e^{-\tau C} (U(u(\tau, \varphi)) - V(v(\tau, \varphi)))] \equiv w(\tau, \varphi) \quad (5)$$

of the sphere K into L . This mapping, generally speaking, is neither regular nor one-to-one. Denote by Σ_τ the image of the sphere K under the mapping—of mapping (5). It is easy to see that the vector φ is the normal to the surface Σ_τ at the point $w(\tau, \varphi)$. We shall assume that the following is satisfied.

Condition b). The surface Σ_τ is locally convex, and the vector φ is the outward normal to Σ_τ at the corresponding point.

Finally, we shall assume that one more condition is satisfied.

Condition c). The orthogonal projection of the subspace CM onto L coincides with all of L .

Then the following holds.

Theorem. *If conditions a), b), c) are satisfied, the linear differential game (1) can be terminated on some set $A \subset R$ in a completely definite computable time $T(z_0)$.*

Proof is carried out by reduction to Theorem 1 of [1], which concerns general, and not only linear, differential games. The set A and the termination time of the game are determined according to the prescription indicated in [1]. Here we shall carry out the reduction itself, and also indicate the condition under which $A = R$.

The function $\omega(s)$ (see [1]) for the linear game (1) is written explicitly, namely

$$z = \omega(s) = \omega(\tau, \varphi, \xi) = e^{\tau C} \left[\xi + \int_0^\tau e^{-rC} (U(u(r, \varphi)) - V(v(r, \varphi))) dr \right], \quad (6)$$

where $s = (s^1, s^2, \dots, s^n)$; $\tau = s^1$; $\varphi = (s^2, \dots, s^n)$ is an arbitrary point of the sphere K ; $\xi = (s^{\nu+1}, \dots, s^n)$ is an arbitrary point of the subspace M ; $u(\tau, \varphi), v(\tau, \varphi)$ are the functions occurring in the mapping (5).

The linear independence of the vectors $\partial\omega/\partial s^2, \dots, \partial\omega/\partial s^n$ is equivalent to the local convexity of the surface Σ_τ , and thus condition 1 of [1] is satisfied in our case. Condition 2 for game (1) can easily be verified in the most general case. In particular, it follows from the stronger condition c). The remaining conditions 3, 4, 6, 8, 9 of [1] for the linear game follow almost automatically from conditions a), b).

We give here the proof that, under the assumptions of the theorem formulated above, the condition of [1] which is most difficult to verify is satisfied, namely condition 5: that for any point s belonging to the upper layer of the mapping $\omega(s)$, the function $H(s)$ is nonnegative. At the final stage of the proof we shall use the following auxiliary proposition, which is contained in § 7 of [1]:

Lemma. *Let the equation*

$$F(\tau, z) = 0, \quad z \notin M, \quad (7)$$

be obtained as a result of eliminating the variables $s^2, \dots, s^n$ from equation (6) (this means that if a point z satisfies equation (6), then the negative number τ satisfies equation (7), and conversely, if the negative number τ satisfies equation (7), then there exists a point $s = (\tau, \varphi, \xi)$ satisfying equation (6)). Then the following two relations hold:

$$\partial F(\tau, \omega(s))/\partial z = -b(s)\varphi e^{-\tau C}; \quad F'_\tau(\tau, \omega(s)) = b(s)H(s), \quad b(s) \neq 0. \quad (8)$$

We construct the function $F(\tau, z)$ for game (1). To this end consider the mapping

$$y = \int_0^\tau w(r, \varphi) dr, \quad (9)$$

obtained by integration with respect to τ of the mapping (5). The image of the sphere K under the mapping (9) is the locally convex surface Δ_τ , and since τ is negative, the vector φ is its inward normal at the corresponding point. It can be proved that Δ_τ is convex as a whole. For simplicity of construction, let us assume that Δ_τ contains the origin of coordinates in its interior. We now construct the auxiliary function $\lambda_\tau(y)$, defined for every point $y \in L$ as follows. Let the ray Oy intersect

surface Δ_τ at the point y_0 . Put

$$\lambda_\tau(y) = |Oy|/|Oy_0|. \quad (10)$$

Then the equation of the surface Δ_τ is written, obviously, in the form $\lambda_\tau(y) - 1 = 0$, and the function $F(\tau, z)$ in the form

$$F(\tau, z) = \lambda_\tau(\pi e^{-\tau C} z) - 1. \quad (11)$$

Since the vector $\partial \lambda_\tau(y)/\partial y$ is the outer normal to the surface Δ_τ , while the vector φ is the inner normal, it follows that $\partial \lambda_\tau(y)/\partial y = -g(y)\varphi$, where $g(y) > 0$. Therefore

$$\frac{\partial F}{\partial z} = \frac{\partial \lambda_\tau(y)}{\partial y} \frac{dy}{dz} = -g(y)\varphi e^{-\tau C}. \quad (12)$$

Comparing this relation with the first of relations (8), we obtain $g(\pi e^{-\tau C} \omega(s)) = b(s)$. Consequently, by virtue of the second relation (8),

$$F'_\tau(\tau, \omega(s)) = g(s)H(s), \quad g(s) > 0. \quad (13)$$

It follows from this that the function $H(s)$ has at each point s the same sign as $F'_\tau(\tau, \omega(s))$.

Let now the point s belong to the upper sheet of the mapping (6), and let $z = \omega(s)$ be its image. Then the number τ is the root of least absolute value of equation (7). On the other hand, from the definition of the function $\lambda_\tau(y)$ it follows that $F(0, z) = +\infty$. Therefore, when passing through the greatest negative root, $F(\tau, z)$ increases and, thus, $F'_\tau(\tau, z) \geq 0$, and hence also $H(s) \geq 0$, as was required to prove.

3. In order that the set A , on which game (1) can be ended, coincide with the whole space R , it is sufficient that, under the mapping ω , every point $z \in R$ have a preimage $s \in S$. By the lemma of the preceding paragraph, for this it is, in turn, sufficient that for any $z \in R$ equation (7) have at least one negative root.

It is not hard to see that this condition is fulfilled if, for each point $z \in R$, there exists at least one negative value τ such that the point $\pi e^{-\tau C} z$ lies inside the surface Δ_τ . Verification of this property can easily be carried out by letting τ tend to $-\infty$.

4. In proving the theorem of item 2, we in fact used the local convexity of the surface Δ_τ , which follows directly from condition b). However, even in the case when the surface Δ_τ is not locally convex, a considerable part of the construction can be preserved. Namely, if the surfaces

$$y = \pi \int_0^\tau e^{-rC} U(u(r, \varphi)) dr, \quad y = \pi \int_0^\tau e^{-rC} V(v(r, \varphi)) dr$$

possess the property that the second of them can be carried by a translation into the interior of the first, then from the surface Δ_τ one can extract a convex, piecewise locally convex part Δ_τ^* , bounding a certain convex domain in L . The surface Δ_τ^* can then be taken as the basis for constructing the function $\lambda_\tau(y)$.

The extraction of the convex part Δ_τ^* of the surface Δ_τ means that, in considering game (1), we use not the whole manifold S , but only a certain part S^* of it, which is obtained if one takes those points (τ, φ, ξ) for which

$$\int_0^\tau w(r, \varphi) dr \in \Delta_\tau^*,$$

and then performs the gluing. However, the question of passage through the gluing surfaces requires special study.

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Academy of Sciences of the USSR

Received
19 I 1967

CITED LITERATURE

1. L. S. Pontryagin, UMN, 21, no. 4 (130), 219 (1966).

Note: Figure translations are in progress. See original paper for figures.

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