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Abstract

Full Text

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HYDROMECHANICS

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ON A SYSTEM OF BOUNDARY-LAYER EQUATIONS FOR AXISYMMETRIC FLOWS

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We shall consider the system of boundary-layer equations for axial stationary and nonstationary flow past bodies of revolution. For these systems, solutions will be constructed below by means of the method of lines; theorems on uniqueness and stability of solutions will be obtained; and the question of the behavior of solutions as time increases without bound will be investigated. In paper ⁽¹⁾, by the method of lines, solutions were constructed for the system of boundary-layer equations for nonstationary axisymmetric and plane-parallel symmetric flows over some finite time interval. Problems concerning continuation of the boundary layer for stationary and nonstationary flows were studied in papers ⁽¹⁻³⁾. We note that the proof of the existence of a solution of the Prandtl system that will be indicated below contains an approximate method for constructing the solution and a proof of its convergence. The method of A. A. Dorodnitsyn for the approximate solution of the system of boundary-layer equations is set forth in ^(4, 5).

1. Stationary axisymmetric boundary layer

The system of boundary-layer equations for an axisymmetric three-dimensional stationary flow of an incompressible fluid in the corresponding coordinates has the form

$$uu_x + vu_y = UU_x + \nu u_{yy}, \quad (ru)_x + (rv)_y = 0 \quad (1)$$

in the domain $D_\theta \{0 \leq x \leq \theta, 0 \leq y < \infty\}$, under the conditions

$$u|_{x=0} = 0, \quad u|_{y=0} = 0, \quad v|_{y=0} = v_0(x), \quad u \rightarrow U(x) \quad \text{as } y \rightarrow \infty. \quad (2)$$

The function $r(x)$ determines the surface of the body being flowed around; $r(0) = 0$; $r_x(0) \neq 0$; U is the velocity of the external flow; ν is the coefficient of viscosity (see ⁽⁶⁾, p. 174).

If one introduces new independent variables

$$\xi = x, \quad \eta = u(x, y)/U(x), \quad (3)$$

then for the function $w = u_y/U$ we obtain the equation

$$\nu w^2 w_{\eta\eta} - \eta U w_\xi + A w_\eta + B w = 0 \quad (4)$$

in the domain $G_\theta \{0 \leq \xi \leq \theta, 0 \leq \eta \leq 1\}$, with conditions

$$w|_{\eta=1} = 0, \quad (\nu w w_\eta - v_0 w + C)|_{\eta=0} = 0, \quad (5)$$

where $A = (\eta^2 - 1)U_x$, $B = -\eta U_x + \eta r_x U/r$, $C = U_x$.

Using the method of lines, we shall prove, under certain natural assumptions, the existence and uniqueness of a solution of problem (4), (5), which, through the transformation (3), leads to a solution of problem (1), (2). Let $f^k(\eta)$ denote $f(kh, \eta)$; $h = \text{const} > 0$; $k = 0, 1, 2, \dots$. We replace equation (4) with conditions (5) by the system of ordinary differential equations

$$\nu(w^k)^2 w_{\eta\eta}^k - \eta U^k (w^k - w^{k-1})/h + A^k w_\eta^k + B^k w^k = 0, \quad (6)$$

$0 \leq \eta \leq 1$; $k = 0, 1, \dots, [\theta/h]$, with the conditions

$$w^k(1) = 0, \quad (\nu w^k w_\eta^k - v_0^k w^k + C^k)|_{\eta=0} = 0. \quad (7)$$

We shall assume that U and r are twice continuously differentiable functions; $U(x) > 0$, $r(x) > 0$ for $x > 0$; $U = ax + b(x)$, where $a = \text{const} > 0$; $|b| \leq K_1 x^2$; v_0, v_{0x}, B_ξ are bounded, $U_x > 0$. It is obvious that $|B| \leq K_2 \xi$; $K_1, K_2 = \text{const}$.

Lemma 1. The system (6), (7) has a solution $w^k(\eta)$, continuous for $0 \leq \eta \leq 1$ and possessing all derivatives for $\eta < 1$.

Lemma 2. For the solution of the problem (6), (7) the estimate

$$M(1-\eta)[|\ln(1-\eta)\beta_1|^{1/2} - M_0]e^{-\gamma_1 kh} \leq w^k(\eta) \leq M(1-\eta)|\ln(1-\eta)\beta_2|^{1/2}e^{\gamma_2 kh}, \quad (8)$$

holds, where $M^2 = 4a$; $\gamma_1, \gamma_2, M_0, \beta_1, \beta_2$ are certain positive constants independent of h , and $|\ln(1-\eta)\beta_1|^{1/2} - M_0 > 0$.

Lemma 3. For $kh \leq x_0$, where $x_0 > 0$ depends on U, r, v_0 , the following estimates are valid for the solution $w^k(\eta)$ of the system (6), (7):

$$-M_1 |\ln(1-\eta)\beta_2|^{1/2} \leq w_\eta^k(\eta) \leq -M_2 |\ln(1-\eta)\beta_1|^{1/2}, \quad (9)$$

$$|(w^k - w^{k-1})/h| \leq M_3(1-\eta) |\ln(1-\eta)\beta_2|^{1/2}, \quad |w^k w_{\eta\eta}^k| \leq M_4, \quad (10)$$

where the constants M_i are positive and do not depend on h .

Estimates for $w^k = z^k$ and $(w^k - w^{k-1})/h = r^k$ can be obtained by considering successively the equations

$$\nu(w^k)^2 z_\eta^k + A^k z^k = \eta U^k r^k - B^k w^k, \quad (11)$$

$$\nu(w^{k+1})^2 r_{\eta\eta}^{k+1} - \eta U^k (r^{k+1} - r^k)/h + A^{k+1} r_\eta^{k+1} + r^{k+1} [B^{k+1} - \eta(U^{k+1} - U^k)]/h +$$

$$+ (w^{k+1} + w^k)(\eta U^k r^k - A^k z^k - B^k w^k)/(w^k)^2 + w_\eta^k (A^{k+1} - A^k)/h + w^k (B^{k+1} - B^k)/h = 0 \quad (12)$$

for $k = 0$, then for $k = 1, 2, \dots, [\theta/h]$, and taking into account the boundary conditions: $r^{k+1}(1) = 0$; $\nu r_\eta^{k+1} - C^{k+1} r^{k+1}/w^k w^{k+1} + (C^{k+1} - C^k)/w^k h + (v_0^{k+1} - v_0^k)/h = 0$ at $\eta = 0$; $\nu z^k - v_0 + C^k/w^k = 0$ at $\eta = 0$, as well as the inequalities (8), from which estimates of the form (9) follow for z^k for some sequence of points η_n tending to $\eta = 1$ as $n \rightarrow \infty$. Estimate (10) for $w^k w_{\eta\eta}^k$ follows directly from equations (6) and the preceding inequalities. By Lemmas 1 and 2, the functions w^k form a compact family in the sense of uniform convergence. The limiting function w as $h \rightarrow 0$ is continuous in G_{x_0} and has a bounded derivative w_ξ ; the derivatives w_η , $w_{\eta\eta}$ are such that ww_η , $ww_{\eta\eta}$ are bounded. Since at the interior points of G_{x_0} equation (4) is parabolic, the derivatives w_ξ , w_η , $w_{\eta\eta}$ satisfy the Hölder condition in any interior subdomain G_{x_0} , and within G_{x_0} equation (4) is valid for w . By virtue of the boundedness of $w_{\eta\eta}^k$ for $\eta < 1 - \delta$ and any $\delta > 0$, w_η satisfies a Lipschitz condition in η for $\eta < 1 - \delta$, and condition (5) is fulfilled for $\eta = 0$. Hence we obtain the following assertion.

Theorem 1. If $\theta \leq x_0$, then in D_θ there exists a solution of the problem (1), (2) possessing the following properties: u, u_y are continuous and bounded in D_θ ; u_{yy}, u_x, v_y are bounded and continuous in y in D_θ and continuous in x at the interior points of D_θ ; v is continuous in y in D_θ , in x at the interior points of D_θ , and bounded for bounded y ; $u_y \rightarrow 0$ as $y \rightarrow \infty$, $u_y > 0$ for $0 \leq y < \infty$; u_{yyy} is bounded in D_θ ; u_{yx} is bounded in D_θ for bounded y ; u_{yx}, u_{yyy} are continuous at the interior points of D_θ . The solution of the problem (1), (2) with the indicated properties is unique.

2. Nonstationary axisymmetric boundary layer. The Prandtl system for an axisymmetric three-dimensional un-

of the steady flow (see (6))

$$u_t + uu_x + vu_y = UU_x + U_t + \nu u_{yy}, \quad (ru)_x + (rv)_y = 0 \quad (13)$$

in the domain $Q_\theta\{0 \leq t < \infty, 0 \leq x \leq \theta, 0 \leq y < \infty\}$, with the conditions

$$u|_{t=0} = u_0(x, y), \quad u|_{x=0} = 0, \quad u|_{y=0} = 0, \quad v|_{y=0} = v_0(t, x),$$

$$u \rightarrow U(t, x) \quad \text{as } y \rightarrow \infty, \quad (14)$$

like problem (1), (2), by means of the transformation of the independent variables

$$\tau = t, \quad \xi = x, \quad \eta = u(t, x, y)/U(t, x) \quad (15)$$

and the introduction of the new function $w = u_y/U$, is reduced to the equation

$$\nu w^2 w_{\eta\eta} - w_\tau - \eta U w_\xi + \tilde{A} w_\eta + \tilde{B} w = 0 \quad (16)$$

in the domain $\Omega_\theta\{0 \leq \tau < \infty, 0 \leq \xi \leq \theta, 0 \leq \eta \leq 1\}$, with the conditions

$$w|_{\tau=0} = u_{0y}/U(0, x) \equiv w_0, \quad w|_{\eta=1} = 0, \quad (\nu w w_\eta - v_0 w + \tilde{C})|_{\eta=1} = 0, \quad (17)$$

where

$$\tilde{A} = (\eta^2 - 1)U_x + (\eta - 1)U_t/U, \quad \tilde{B} = \eta r_{xU}/r - U_t/U - \eta U_x,$$

$$\tilde{C} = U_x + U_t/U.$$

We shall assume that $r(t, x)$ and $U(t, x)$ are functions twice continuously differentiable in Q_θ ; $U > 0$ and $r > 0$ for $x > 0$; $U = ax + b(t, x)$, where $a = \text{const} > 0$, $|b| \leq K_3 x^2$, $K_3 = \text{const}$; $r(t, 0) = 0$, $r_x(t, 0) \neq 0$; $v_0(t, 0) = \text{const}$; the functions U_x , U_t/U , r_{xU}/r , v_0 and their derivatives with respect to t and x are bounded in Q_θ , and the derivatives of these functions with respect to t , as well as U_t/U , do not exceed in absolute value $K_4 x$, where $K_4 = \text{const}$.

Let $f^{mk}(\eta) \equiv f(mh, kh, \eta)$. We replace equation (16) in the domain Ω_θ , with conditions (17), by a system of ordinary differential equations

$$\nu(w^{mk})^2 w_{\eta\eta}^{mk} - \frac{w^{mk} - w^{m-1}k}{h} - \eta U^{mk} \frac{w^{mk} - w^{mk-1}}{h} + \tilde{A}^{mk} w_{\eta}^{mk} + \tilde{B}^{mk} w^{mk} = 0, \quad (18)$$

$$0 \leq \eta \leq 1; \quad m = 1, 2, \dots; \quad k = 0, 1, \dots, [\theta/h]; \quad h = \text{const} > 0,$$

with the conditions

$$w^{mk}(1) = 0, \quad (\nu w^{mk} w_{\eta}^{mk} - v_0^{mk} w^{mk} + \tilde{C}^{mk})|_{\eta=0} = 0, \quad w^{0k} = w_0(kh, \eta). \quad (19)$$

With respect to the function $w_0(\xi, \eta)$ we shall assume the following: $w_0 \leq M(1-\eta)|\ln(1-\eta)\alpha_1|^{1/2}e^{\gamma\xi}$, where $M^2 = 4a$; $w_0 \geq K_4(1-\eta)|\ln(1-\eta)\alpha_2|^{1/2}e^{-\gamma\xi}$, moreover $K_4^2 \geq 2a$, $|w_{0\xi}| \leq K_5(1-\eta)|\ln(1-\eta)\alpha_2|^{1/2}$, if $|\tilde{B}_{\xi}| \leq K_6$, and K_4 is an arbitrary sufficiently small constant, while $|w_{0\xi}| \leq K_7(1-\eta)\xi|\ln(1-\eta)\alpha_2|^{1/2}$, if $|\tilde{A}_{\xi}| + |\tilde{B}_{\xi}| + |\tilde{C}_{\xi}| \leq K_8\xi$; let, in addition, $w_{0\eta} \leq 0$, $w_{0\eta} \geq -K_9|\ln(1-\eta)\alpha_1|^{1/2}$,

$$|\nu w_0^2 w_{0\eta\eta} - \eta U(0, \xi) w_{0\xi} + \tilde{A}(0, \xi, \eta) w_{0\eta} + \tilde{B}(0, \xi, \eta) w_0| \leq K_{10}\xi(1-\eta)|\ln(1-\eta)\alpha_2|^{1/2}$$

and the compatibility condition is fulfilled

$$\nu w_0(\xi, 0) w_{0\eta}(\xi, 0) - v_0(0, \xi) w_0(\xi, 0) + \tilde{C}(0, \xi) \equiv \nu u_{0yy}(x, 0) + (UU_x + U_t)|_{t=0} - v_0(0, x) u_{0y}(x, 0) = 0.$$

Here K_i , α_i , γ are certain positive constants. Obviously, all these conditions on w_0 will be satisfied if, for example, $u_0(x, y)$ is a solution of problem (1), (2), or coincides with it in a neighborhood of the straight line $x = 0$, and also for large y , with U , r , v_0 taken at $t = 0$. This means that w_0 is a certain perturbation of the velocity u in the steady boundary layer.

Lemma 4. *The system (18) with conditions (19) has a solution w^{mk} , continuous for $0 \leq \eta \leq 1$ and possessing all continuous derivatives for $\eta < 1$. For $kh \leq \tilde{x}_0$ and $0 \leq mk < \infty$, the following estimates hold for w^{mk} :*

$$M_5(1-\eta)|\ln(1-\eta)\alpha_4|^{1/2} \leq w^{mk} \leq M(1-\eta)|\ln(1-\eta)\alpha_3|^{1/2}e^{\gamma kh},$$

$$0 \geq w_{\eta}^{mk} \geq -M_6|\ln(1-\eta)\alpha_3|^{1/2}, \quad \left| \frac{w^{mk} - w^{m-1}k}{h} \right| \leq$$

$$\leq M_7 k h (1 - \eta) |\ln(1 - \eta) \alpha_4|^{1/2}, \quad (20)$$

$$\left| \frac{w^{mk} - w^{mk-1}}{h} \right| \leq M_8 (1 - \eta) |\ln(1 - \eta) \alpha_4|^{1/2}.$$

Here M_i and α_i are certain positive constants independent of h .

Theorem 2. Let U, r, v_0, u_0 satisfy the conditions indicated above in Sec. 2. Then in the domain Q_θ , for $\theta \leq \tilde{x}_0$, there exists a solution of problem (13), (14) having the following properties: u, u_y are continuous and bounded in Q_θ ; u_{yy}, u_x, u_t, v_y are bounded and continuous with respect to y in Q_θ ; v is continuous with respect to y and bounded for bounded y ; $u_y \rightarrow 0$ as $y \rightarrow \infty$, $u_y > 0$ for $y \geq 0$; u_{yyy} is bounded in Q_θ , u_{yx}, u_{yt} are bounded in Q_θ for bounded y . The solution of problem (13), (14) possessing these properties is unique.

3. Stability. Denote by $\tilde{u}(t, x, y)$ the solution of problem (13), (14) for

$$U = \tilde{U}(t, x), \quad r = \tilde{r}(t, x), \quad v_0 = \tilde{v}_0(t, x)$$

and $u_0(x, y)$, and by $u(x, y)$ the solution of problem (1), (2) corresponding to the functions $U(x), r(x), v_0(x)$. Suppose that these functions satisfy the conditions of Theorems 2 and 1, respectively.

Theorem 3. Let the functions

$$\tilde{U} - U, \quad \tilde{U}_x - U, \quad \tilde{U}_t / \tilde{U}, \quad \tilde{v}_0 - v_0, \quad \tilde{r}_x \tilde{U} / \tilde{r} - r_{xU} / r \quad (21)$$

tend to zero as $t \rightarrow \infty$ uniformly in D_θ . Then for any $\delta > 0$ there exists a constant $C_1(\delta)$ such that in D_θ

$$|\tilde{u}(t, x, y) / \tilde{U}(t, x) - u(x, y) / U(x)| \leq \delta + C_1(\delta) e^{-\sigma t},$$

where $\sigma = \text{const} > 0$ depends on the data of problem (1), (2), $\theta = \min(x_0, \tilde{x}_0)$.

Theorem 4. If the functions (21) are equal to zero for $t \geq t_0 > 0$, where $t_0 = \text{const} < \infty$, then in D_θ , for $y \leq y_0$,

$$|\tilde{u}(t, x, y) - u(x, y)| \leq C_2 U(x) e^{-\sigma t},$$

where the constant C_2 depends on y_0 , $\theta = \min(x_0, \tilde{x}_0)$.

Theorem 5. Let the functions (21) not exceed $\varepsilon > 0$ in absolute value, and let

$$|\tilde{w}(0, \xi, \eta) - w(\xi, \eta)| \leq \varepsilon,$$

where w is the solution of problem (4), (5), and $\tilde{w}$ is the solution of problem (16), (17), corresponding to the functions u and $\tilde{u}$. Then

$$|\tilde{u}(t, x, y) / \tilde{U}(t, x) - u(x, y) / U(x)| \leq C_3 \varepsilon$$

for all t in D_θ , where the constant C_3 depends on the data of problem (1), (2) and on the constants entering into (20).

Let us note that, by an analogous method, one can investigate the boundary-layer equations for plane-parallel symmetric flows of an incompressible fluid. The question of the stability of solutions of the problem of the continuation of the boundary layer was considered in work (7).

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