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Abstract

Full Text

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MATHEMATICS

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ON THE COEFFICIENTS OF UNIVALENT FUNCTIONS

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Let S_p be the class of functions

$$f_p(z) = z + C_2^{(p)} z^{p+1} + \dots$$

regular and univalent in the disk $|z| < 1$, with p -fold rotational symmetry ($p = 1, 2, \dots$; $S_1 = S$), and let Σ be the class of functions

$$F(z) = z + a_0 + a_1 z^{-1} + \dots,$$

regular and univalent in the domain $1 < |z| < \infty$. Denote:

$$\ln \frac{f(z)}{z} = \sum_{k=1}^{\infty} 2\gamma_k z^k, \quad f(z) \in S. \quad (1)$$

Each function $F(z) \in \Sigma$ generates a system of functions $\{A_n(z)\}$ ($n = 1, 2, \dots$) by means of the expansion:

$$\ln \frac{z-t}{F(z)-F(t)} = \sum_{n=1}^{\infty} A_n(t) z^{-n}, \quad |z| > 1, |t| > 1. \quad (2)$$

Among the properties of the system $\{A_n(z)\}$ ⁽¹⁾ we recall the inequality

$$\sum_{n=1}^{\infty} n |A_n(z)|^2 \leq \ln \frac{1}{1-r^2}, \quad |z| = \frac{1}{r} > 1. \quad (3)$$

Below, using this inequality, we derive estimates for certain mean quantities for the coefficients γ_k and C_k^p .

Theorem 1. For a function $f(z) \in S$, for every n ($n = 1, 2, \dots$) the inequality

$$\sum_{k=1}^n k |\gamma_k|^2 \leq \sum_{k=1}^n \frac{1}{k} + \delta \quad (4)$$

holds, where δ is an absolute constant for which the estimate $\delta < 0.312$ holds.

Proof. From a function $f(\zeta) \in S$ construct

$$F(z) = \frac{1}{f(\zeta)} \in \Sigma, \quad z = \frac{1}{\zeta},$$

and then, for an arbitrary finite w , consider the function

$$\ln \frac{z}{F(z) - w}$$

and its Taylor expansion about $z = \infty$. Let

$$\ln \frac{z}{F(z) - w} = \sum_{k=1}^{\infty} Q_k(w) z^{-k}. \quad (5)$$

If in equality (5) we put $w = 0$ and compare the resulting equality with expansion (1), then we shall have:

$$2\gamma_k = Q_k(0) \quad (k = 1, 2, \dots). \quad (6)$$

Since the function $F(z)$ does not vanish in the domain $|z| > 1$, the interior of any level line C_ρ , $\rho > 1$, contains the point $w = 0$, and then, by the maximum principle for subharmonic functions, taking (6) into account, we obtain

$$4 \sum_{k=1}^n k |\gamma_k|^2 = \sum_{k=1}^n k |Q_k(0)|^2 \leq \max_{w \in C_\rho} \sum_{k=1}^n k |Q_k(w)|^2. \quad (7)$$

(the subharmonicity of the function $\sum_{k=1}^n k |a_k(w)|^2$ was used earlier by Pommerenke [2] in estimating Faber polynomials). But on the level line C_ρ , $Q_k(w) = Q_k(F(z))$, $|z| = \rho$, and the quantity $Q_k(F(z))$ is related to the function $A_k(z)$ by the relation (1)

$$Q_k(F(z)) = \frac{1}{2} z^k + A_k(z) \quad (k = 1, 2, \dots). \quad (8)$$

Now from (7), (8), and (3) we find:

$$\begin{aligned} \sum_{k=1}^n k |\gamma_k|^2 &\leq \frac{1}{4} \max_{|z|=\rho} \sum_{k=1}^n k \left| \frac{1}{k} z^k + A_k(z) \right|^2 \leq \\ &\leq \frac{1}{2} \left(\sum_{k=1}^n \frac{1}{k} \rho^{2k} + \ln \frac{1}{1 - \rho^2} \right), \quad \rho = \frac{1}{r} > 1. \end{aligned}$$

Choosing $\rho = 2^{1/(2n+1)}$ and carrying out the necessary transformations, we arrive at the conclusion of the theorem.

Corollary. For the function

$$f_p(z) = \sum_{n=0}^{\infty} C_{n+1}^{(p)} z^{np+1} \in S_p \quad (p = 1, 2, \dots)$$

we have

$$\frac{1}{n} \sum_{k=1}^n |C_k^{(p)}| \leq A n^{2/p-1} \quad (n = 1, 2, \dots), \quad (9)$$

where $A < e^{(\delta+2c)/p}$ (c is Euler's constant).

Proof. If $f_p(z) \in S_p$, then $f_p^p(z^{1/p}) = f(z) \in S$. Therefore we have the identity

$$\ln \frac{f_p(z^{1/p})}{z^{1/p}} = \frac{1}{p} \ln \frac{f(z)}{z} = \sum_{k=1}^{\infty} \frac{2}{p} \gamma_k z^k,$$

or, after exponentiation,

$$\exp \left[\sum_{k=1}^{\infty} \frac{2}{p} \gamma_k z^k \right] = \sum_{k=0}^{\infty} C_{k+1}^{(p)} z^k. \quad (10)$$

From (10) it follows that

$$\sum_{k=0}^{n-1} |C_{k+1}^{(p)}| \leq \exp \left[\frac{2}{p} \sum_{k=1}^{n-1} |\gamma_k| \right] \quad (n = 2, 3, \dots).$$

Using Cauchy's inequality and Theorem 1, we have

$$\sum_{k=1}^{n-1} |\gamma_k| \leq \left[\sum_{k=1}^{n-1} k |\gamma_k|^2 \cdot \sum_{k=1}^{n-1} \frac{1}{k} \right]^{1/2} < \sum_{k=1}^{n-1} \frac{1}{k} + \frac{\delta}{2} < \ln n + \frac{\delta + 2c}{2}.$$

From the last two inequalities we obtain the desired assertion.

Remark. It is interesting to compare inequality (9) and the Segre conjecture

$$|C_n^{(p)}| = O(n^{2/p} - 1) \quad (p = 1, 2, \dots), \quad (11)$$

which was disproved by Littlewood for large p [3].

Lemma 1. Let $\{A_k\}_1^\infty$ be an arbitrary sequence of complex numbers generating the sequence $\{D_k\}_0^\infty$ by means of the formal expansion:

$$\exp \left[\sum_{k=1}^{\infty} A_k z^k \right] = \sum_{k=0}^{\infty} D_k z^k. \quad (12)$$

Let

$$\frac{1}{(1-z)^\lambda} = \sum_{k=0}^{\infty} d_k z^k, \quad d_k = d_k(\lambda), \quad \lambda > 0. \quad (13)$$

Then

$$\begin{aligned} \frac{1}{n} \sum_{k=0}^{n-1} \frac{|D_k|^2}{d_k} &\leq \frac{d_n}{\lambda} \exp \left\{ \frac{\lambda}{n d_n} \sum_{\nu=1}^{n-1} [(n-\nu) d_{n-\nu} - (n-\nu-1) d_{n-\nu-1}] \times \right. \\ &\quad \left. \times \left[\frac{1}{\lambda^2} \sum_{k=1}^{\nu} k |A_k|^2 - \sum_{k=1}^{\nu} \frac{1}{k} \right] \right\} \quad (n = 2, 3, \dots). \end{aligned} \quad (14)$$

The equality sign in (14) holds if and only if

$$A_k = \frac{\lambda}{k} \eta^k \quad (k = 1, 2, \dots, n-1), \quad |\eta| = 1.$$

Lemma 2. For any sequence of complex numbers $\{A_k\}$ ($k = 1, 2, \dots$), for each n ($n = 1, 2, \dots$) the inequality

$$|D_n| \leq \exp \left[\frac{1}{2} \left(\sum_{k=1}^n k |A_k|^2 - \sum_{k=1}^n \frac{1}{k} \right) \right] \quad (15)$$

holds (the notation (12) is used).

The equality sign in (15) holds if and only if

$$A_k = (1/k) \eta^k \quad (k = 1, 2, \dots, n), \quad |\eta| = 1.$$

Inequalities (14) and (15) were obtained by the author jointly with N. A. Lebedev.

Theorem 2. For a function

$$f_p(z) = \sum_{k=0}^{\infty} C_{k+1}^{(p)} z^{kp+1} \in S_p \quad (p = 1, 2, \dots)$$

we have

$$\frac{1}{n} \sum_{k=1}^n \frac{|C_k^{(p)}|^2}{C_k^{(p)*}} \leq \frac{p}{2} C_{n+1}^{(p)*} \exp \left[\frac{2\delta}{p} \right] < \exp \left[\frac{2}{p} (\delta + C) \right] n^{2/p-1}, \quad (16)$$

where $C_k^{(p)*}$ ($k = 1, 2, \dots$) are the coefficients in the expansion of the function $f_p^*(z) = z/(1-z^p)^{2/p} \in S_p$, δ is defined in Theorem 1, and C is Euler's constant.

Proof. For any function $f_p(z) \in S_p$, as proved, identity (10) holds. If we now put $(2/p)\gamma_k = A_k$ ($k = 1, 2, \dots$), $C_{k+1}^{(p)} = D_k$ ($k = 0, 1, \dots$), set $\lambda = 2/p$, and apply Lemma 1 to the sequence $\{A_k\}$, then we obtain:

$$\begin{aligned} \frac{1}{n} \sum_{k=0}^{n-1} \frac{|C_{k+1}^{(p)}|^2}{d_k} &\leq \frac{d_n}{\lambda} \exp \left\{ \frac{\lambda}{nd_n} \sum_{\nu=1}^{n-1} [(n-\nu)d_{n-\nu} - (n-\nu-1)d_{n-\nu-1}] \times \right. \\ &\quad \left. \times \left[\sum_{k=1}^{\nu} k|\gamma_k|^2 - \sum_{k=1}^{\nu} \frac{1}{k} \right] \right\} \quad (n = 2, 3, \dots). \end{aligned} \quad (17)$$

By Theorem 1, the differences

$$\sum_{k=1}^{\nu} k|\gamma_k|^2 - \sum_{k=1}^{\nu} \frac{1}{k} \leq \delta$$

for every $\nu \geq 1$, and since the coefficients of these differences are positive, the exponent in (22) does not exceed

$$\frac{\lambda}{nd_n} (n-1)d_{n-1}\delta.$$

Taking this into account, and also the identity

$$(n-1)d_{n-1}(\lambda)/nd_n(\lambda) = (n-1)/(\lambda+n-1), \quad (18)$$

and noting that $d_k(2/p) = C_{k+1}^{(p)*}$ ($k = 0, 1, \dots$), from (17) we obtain the first part of relation (16). The second part of (16) is obtained elementarily. The theorem is proved.

Remark. It is easy to see that the left-hand side of inequality (16) for the function $f_p^*(z)$ is equal to $\frac{p}{2} C_{n+1}^{(p)*}$, and consequently, in the class S_p ($p = 1, 2, \dots$)

the estimate (16) for the quantity $\frac{1}{n} \sum_{k=1}^n \left| \frac{C_k^{(p)}}{C_k^{(p)*}} \right|^2$ is no more than $e^{2\delta \cdot p}$ times greater than the exact one. We note that inequality (9) follows from inequality (16).

Theorem 3. For the coefficients of the function

$$f_2(z) = \sum_{n=0}^{\infty} C_{n+1}^{(2)} z^{2n+1} \in S_2$$

the estimate

$$|C_n^{(2)}| < 1.17 \quad (n = 2, 3, \dots) \quad (19)$$

holds.

Proof. Identity (10), for $p = 2$, is written in the form

$$\exp \left[\sum_{k=1}^{\infty} \gamma_k z^k \right] = \sum_{k=0}^{\infty} C_{k+1}^{(2)} z^k. \quad (20)$$

Now we apply Lemma 2 to the sequence $\{\gamma_k\}$ ($k = 1, 2, \dots$). Then we obtain the inequality:

$$|C_{n+1}^{(2)}| \leq \exp \left[\frac{1}{2} \left(\sum_{k=1}^n k |\gamma_k|^2 - \sum_{k=1}^n \frac{1}{k} \right) \right] \quad (n = 1, 2, \dots), \quad (21)$$

which, taking Theorem 1 into account, leads to the estimate

$$|C_n^{(2)}| \leq e^{\frac{1}{2}\delta} < 1.17.$$

The theorem is proved.

The uniform boundedness of the coefficients in the class S_2 (i.e. the inequality $|C_n^{(2)}| \leq A$) was proved by Littlewood and Paley ⁽⁴⁾. Fekete and Szegő ⁽⁵⁾ showed that $A > 1$. The smallest of the obtained values is $A = 2.54 \dots$ ⁽⁶⁾.

Since for each $n > 2$

$$\sup_{f_2(z) \in S_2} |C_n^{(2)}| = A_n > 1$$

⁽⁵⁾, the relation obtained from (21) and (4) is of interest:

$$\sum_{k=1}^n \frac{1}{k} + 2 \ln A_{n+1} \leq \sup_{f(z) \in S} \sum_{k=1}^n k |\gamma_k|^2 \leq \sum_{k=1}^n \frac{1}{k} + \delta.$$

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Note: Figure translations are in progress. See original paper for figures.

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