

GENERAL INTEGRAL REPRESENTATIONS OF HOLOMORPHIC FUNCTIONS

I. I. Bavrin

1967

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196701.95500>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.53:517.55

MATHEMATICS

I. I. Bavrín

GENERAL INTEGRAL REPRESENTATIONS OF HOLOMORPHIC FUNCTIONS

(Presented by Academician M. A. Lavrent'ev, 22 IV 1966)

In the present note, in the case of several complex variables, a general integral formula is established for functions holomorphic in convex complete n -circular domains of the space C^n , $n \geq 2$ (Theorem 4), and a general integral formula for functions holomorphic in convex domains of the space C^n , $n \geq 2^*$ (Theorem 5). Each of these two formulas, for every value of k from $\{1, 2, \dots, \mu\}$, includes an infinite set of integral representations. In formulating Theorems 4 and 5 we adhere to the definitions and notation used in ⁽¹⁾. One of the auxiliary formulas is the general integral formula obtained here in the case of one complex variable (Theorem 3) for functions holomorphic in star-shaped domains.

1. Let G be a star-shaped** domain with respect to the origin in the space C^n of complex variables $z_1, \dots, z_n$, $n \geq 1$, and let the function $F(z_1, \dots, z_n)$ be holomorphic in G . Further, let γ be an arbitrary positive number; p, q natural numbers with $p \geq q$, and let $\gamma_p, \gamma_{p-1}, \dots, \gamma_q, \gamma_0$ be arbitrary positive numbers. Introduce the notation (for brevity, instead of $F(z_1, \dots, z_n)$ we write F)

$$L_\gamma[F] = \gamma F + \sum_{\nu=1}^n z_\nu F'_{z_\nu}, \quad L_{q,p}^{(p-q+1)}[F] = L_p[L_{p-1} \dots [L_q[F]] \dots],$$

$$L_{\left(\begin{smallmatrix} p-q+1 \\ \gamma_q \\ \gamma_p \end{smallmatrix}\right)}[F] = L_{\gamma_p}[L_{\gamma_{p-1}} \dots [L_{\gamma_q}[F]] \dots]$$

and put

$$L_{p,p-1}^{(0)}[F] = F, \quad L_{\left(\begin{smallmatrix} 0 \\ \gamma_p \\ \gamma_{p-1} \end{smallmatrix}\right)}[F] = F \quad (p \geq 1).$$

Theorem 1. *If the function $F(z_1, \dots, z_n)$ ($n \geq 1$) is holomorphic in the domain G , then for every natural k in the domain G we have*

$$F(z_1, \dots, z_n) = \int_0^1 d\varepsilon_1 \dots \int_0^1 d\varepsilon_{k-1} \int_0^1 \varepsilon_1^{\gamma_1-1} \dots \varepsilon_k^{\gamma_k-1} \times \\ \times L_{\left(\begin{smallmatrix} \gamma_1 \\ \gamma_k \end{smallmatrix}\right)}^{(k)} [F(\varepsilon_1 \dots \varepsilon_k z_1, \dots, \varepsilon_1 \dots \varepsilon_k z_n)] d\varepsilon_k, \quad (1)$$

where $\varepsilon_1, \dots, \varepsilon_k$ are real and $\gamma_1, \dots, \gamma_k$ are arbitrary positive numbers satisfying the condition $\gamma_j \geq 1$ ($j = 1, \dots, k$).

For $k = 1$ formula (1) is proved in the same way as formula (1) in the author's note ⁽¹⁾, and then for any natural k by induction.

2. Let G_1 be a star-shaped domain with respect to the origin in the space C^1 , bounded by a closed rectifiable Jordan curve Γ , and

* The same is true in the case of star-shaped domains in C^n , $n \geq 2$.

** A domain star-shaped with respect to the origin is called a domain containing, together with each point, the entire segment connecting this point with the origin.

$F(z)$ is a function holomorphic in the domain G_1 . On the basis of formula (1) ($n = 1$) and Cauchy's formula ($n = 1$) one obtains

Theorem 2. *If the function $F(z)$, holomorphic in the domain G_1 , and all its derivatives up to order μ ($\mu > 0$) inclusive are continuous in the closed domain $\overline{G}_1$, then for $k = 1, \dots, \mu$ and $z \in G_1$ the formula holds*

$$F(z) = \frac{1}{2\pi i} \int_{\Gamma} K_{\left(\begin{smallmatrix} \gamma_1 \\ \gamma_k \end{smallmatrix}\right)}(\xi, z) L_{\left(\begin{smallmatrix} \gamma_1 \\ \gamma_k \end{smallmatrix}\right)}^{(k)} [F(\xi)] d\xi, \quad (2)$$

where

$$K_{\left(\begin{smallmatrix} \gamma_1 \\ \gamma_k \end{smallmatrix}\right)}(\xi, z) = \int_0^1 d\varepsilon_1 \dots \int_0^1 d\varepsilon_{k-1} \int_0^1 \frac{\varepsilon_1^{\gamma_1-1} \dots \varepsilon_k^{\gamma_k-1}}{\xi - \varepsilon_1 \dots \varepsilon_k z} d\varepsilon_k, \quad (3)$$

$\gamma_1, \dots, \gamma_k$ are arbitrary positive numbers satisfying $\gamma_j \geq 1$ ($j = 1, \dots, k$), and integration is performed over the contour Γ in the positive direction.

Let us point out the distinctive features of the integral representation (2): a) it expresses the values of the function $F(z)$ in the domain G_1 through the values of the operator

$$L_{\left(\begin{smallmatrix} \gamma_1 \\ \gamma_k \end{smallmatrix}\right)}^{(k)} [F]$$

on the boundary of the domain G_1 ; b) since $\gamma_1, \dots, \gamma_k$ are arbitrary positive numbers satisfying $\gamma_j \geq 1$ ($j = 1, \dots, k$), formula (2), for each value of k from $\{1, 2, \dots, \mu\}$, includes an infinite set of integral representations.

Putting

$$K_{\left(\begin{smallmatrix} \gamma_1 \\ \gamma_0 \end{smallmatrix}\right)}(\xi, z) = \frac{1}{\xi - z},$$

one can, in the case of the domain G_1 , combine Cauchy's formula and formula (2) into a single formula (2). Thus we obtain the following general theorem 3.

Theorem 3. *If the function $F(z)$, holomorphic in the domain G_1 , and all its derivatives up to order μ ($\mu \geq 0$; $F^{(0)} = F$) inclusive are continuous in the closed domain $\bar{G}_1$, then for $k = 0, 1, \dots, \mu$ and $z \in G_1$ formula (2) holds, where $\gamma_1, \dots, \gamma_k$ are arbitrary positive numbers satisfying $\gamma_j \geq 1$ ($j = 1, \dots, k$).*

3. Let us pass to the case $n \geq 2$. Let B and H be bounded star-shaped domains with respect to the origin in C^n , respectively with piecewise-smooth and smooth boundaries.

Theorem 4. *Let $D \in (T)$, and let the function $f(z)$ ($n \geq 2$) be holomorphic in D , and let α be a number equal to 0 or 1. Then, if the functions*

$$f_{z_\nu^{(\alpha)}}(z), \quad \nu = 1, \dots, n^{**},$$

and all their partial derivatives up to order μ ($\mu \geq 0$) inclusive are continuous

* We note that if the domain G_1 is the disk $|z| < 1$, then under the conditions of theorem 3, for $k = 0, 1, \dots, \mu$ and $|z| < 1$ the formulas

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} P\left(\frac{\gamma_1}{\gamma_k}\right)(\rho, \varphi - \psi) L^{(k)}\left(\frac{\gamma_1}{\gamma_k}\right)[F(e^{i\varphi})] d\varphi \quad (z = \rho e^{i\psi}),$$

$$F(z) = i \operatorname{Im} F(0) + \frac{1}{2\pi} \int_0^{2\pi} S\left(\frac{\gamma_1}{\gamma_k}\right)(e^{i\varphi}, z) \operatorname{Re} L^{(k)}\left(\frac{\gamma_1}{\gamma_k}\right)[F(e^{i\varphi})] d\varphi,$$

hold, where the kernels

$$P\left(\frac{\gamma_1}{\gamma_k}\right)(\rho, \varphi - \psi) \quad \text{and} \quad S\left(\frac{\gamma_1}{\gamma_k}\right)(e^{i\varphi}, z)$$

for $k = 1, \dots, \mu$ have the form obtained from (3) by replacing in (3)

$$\frac{1}{\xi - \varepsilon_1 \dots \varepsilon_k z}$$

respectively by

$$\frac{1 - (\varepsilon_1 \dots \varepsilon_k \rho)^2}{1 + (\varepsilon_1 \dots \varepsilon_k \rho)^2 - 2\varepsilon_1 \dots \varepsilon_k \rho \cos(\varphi - \psi)}$$

and

$$\frac{e^{i\varphi} + \varepsilon_1 \dots \varepsilon_k z}{e^{i\varphi} - \varepsilon_1 \dots \varepsilon_k z},$$

and for $k = 0$ these kernels are, respectively, the Poisson and Schwarz kernels.

** If $\alpha = 0$, then these functions are one and the same function $f(z)$.

are continuous in $D \cup S$, then for $k = 0, 1, \dots, \mu$ and $z \in D$

$$f(z) = \alpha f(0) + \frac{1}{n + \alpha(1 - n)} \sum_{\nu=1}^n \frac{z_{\nu}^{\alpha}}{(2\pi)^{n_i}} \int d\omega_* \int d\omega_{\theta} \int_{|\xi|=1} L_{\alpha+1, n-1}^{(n-1-\alpha)} \left[K \left(\frac{\gamma_1}{\gamma_k} \right) (\xi, \mu) \right] \times \\ \times L \left(\frac{\gamma_1}{\gamma_k} \right)^{(k)} \left[F_{0z_{\nu}^{\alpha}}^{(\alpha)} (\xi, r, \theta) \right] d\xi, \quad (4)$$

where $\gamma_1, \dots, \gamma_k$ are arbitrary positive numbers satisfying $\gamma_j \geq 1$ ($j = 1, \dots, k$).*

Theorem 5. Let the function $F(z_1, \dots, z_n)$ ($n \geq 2$) be holomorphic in the domain D^{**} , and let α be a number equal to 0 or 1. Then, if the functions $F_{z_{\nu}^{\alpha}}^{(\alpha)}(z_1, \dots, z_n)$, $\nu = 1, \dots, n$, and all their partial derivatives up to order μ ($\mu \geq 0$) inclusive are continuous in the closed domain $\overline{D}$, then for $k = 0, 1, \dots, \mu$ and points $(z_1, \dots, z_n) \in D$ there holds a formula (in view of the brevity of the presentation we do not write it out here), analogous in character to formula (4), expressing the values of the function $F(z_1, \dots, z_n)$ in the domain D through the values

$$L \left(\frac{\gamma_1}{\gamma_k} \right)^{(k)} \left[F_{z_{\nu}^{\alpha}}^{(\alpha)} \right],$$

where $\gamma_1, \dots, \gamma_k$ are arbitrary positive numbers satisfying $\gamma_j \geq 1$ ($j = 1, \dots, k$), on the boundary of the domain D , up to the summand $\alpha F(0, \dots, 0)$. Similarly in the case of the domains B and H^{***} .

For the proof of Theorem 4 one should use Theorem 3 of this note and the integral formula (for $k = 0$) in Theorem 2 from ⁽¹⁾; for the proof of Theorem 5 in the case of the domain D , formula (1) of the present note, formula (2) from ⁽¹⁾, and formula (2) from ⁽²⁾; while in the case of the domains B and H the only difference is that instead of formula (2) from ⁽²⁾ one uses, respectively, the Martinelli-Bochner integral representation (see, for example, ⁽³⁾) and integral formula (2) from ⁽⁴⁾.

Remark 1. Each of the four integral formulas in Theorems 4 and 5 (in Theorem 5 three formulas are meant: one in the case of the domain D , another for the domain B , and a third for H) for any value of k from $\{1, 2, \dots, \mu\}$ includes an infinite set of integral representations, since $\gamma_1, \dots, \gamma_k$ are arbitrary positive numbers satisfying $\gamma_j \geq 1$ ($j = 1, \dots, k$).

4. Formula (1) ($k = 1$) remains valid also in the case when γ_1 is any positive number. But for $0 < \gamma_1 < 1$ the integral entering this formula should be understood as an improper one. Taking into account the same remark concerning analogous integrals, Theorem 1 and all the other results of this note connected with $\gamma_1, \dots, \gamma_k$ remain valid also in the case when $\gamma_1, \dots, \gamma_k$ are arbitrary positive numbers.

5. With the aid of his integral representations, A. A. Temlyakov^(5,6) obtained integral representations for a certain class of meromorphic functions in the case of two complex variables. In an analogous way, with the aid of Theorem 4 of the present note, integral representations are established for the corresponding class of meromorphic functions of n ($n \geq 2$) complex variables.

Moscow Regional Pedagogical Institute
named after N. K. Krupskaya

Received
27 III 1966

REFERENCES

- ¹ I. I. Bavrin, DAN, **169**, No. 3 (1966).
² L. A. Aizenberg, DAN, **151**, No. 6 (1963).
³ V. S. Vladimirov, *Methods of the Theory of Functions of Several Complex Variables*, Moscow, 1964.
⁴ L. A. Aizenberg, DAN, **155**, No. 1 (1964).
⁵ A. A. Temlyakov, DAN, **143**, No. 5 (1962).
⁶ A. A. Temlyakov, Scientific Notes of the Moscow Regional Pedagogical Institute named after N. K. Krupskaya, **110**, 3 (1962).

* Under the conditions of Theorem 4 there are also two more general formulas corresponding to the formulas indicated in the footnote on p. 1252.

** D here is the same convex domain in C^n that was considered in ^(1,2).

*** The integral formula in the case of the domain H is more general in construction than in the case of the domain B .

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.