

FOURIER SERIES AND MAXIMAL THEOREMS

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Abstract

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MATHEMATICS

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FOURIER SERIES AND MAXIMAL THEOREMS

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Consider a function of two variables $f(x, y)$. Suppose that it is periodic with respect to each of the variables and that $f(x, y) \in L(R)$, where $R = [-\pi, \pi; -\pi, \pi]$. Denote by $\bar{f}_i(x, y)$ ($i = 1, 2, 3$) the conjugate functions of two variables

$$\bar{f}_1(x, y) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x + s, y) \operatorname{ctg} \frac{s}{2} ds,$$

$$\bar{f}_2(x, y) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x, y + t) \operatorname{ctg} \frac{t}{2} dt,$$

$$\bar{f}_3(x, y) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x + s, y + t) \operatorname{ctg} \frac{s}{2} \operatorname{ctg} \frac{t}{2} dt ds.$$

By $\sigma[f]$ we shall denote the double Fourier series of the function $f(x, y)$, and by the symbols $\bar{\sigma}[f; x]$, $\bar{\sigma}[f; y]$, and $\bar{\sigma}[f; x, y]$ the double conjugate trigonometric series to the series $\sigma[f]$, respectively in the variable x , in the variable y , and in the totality of the variables x and y . Further, let $\sigma_{mn}^{(i, \alpha, \beta)}(x, y)$ ($i = 0, 1, 2, 3$), $\alpha, \beta > 0$, be the Cesàro $(C; \alpha, \beta)$ -means respectively of the series $\sigma[f]$, $\bar{\sigma}[f; x]$, $\bar{\sigma}[f; y]$, and put

$$\varphi_i(x, y) = \sup_{m, n \geq 0} |\sigma_{mn}^{(i, \alpha, \beta)}(x, y)| \quad (i = 0, 1, 2, 3),$$

$$\varphi_4(x, y) = \sup_{0 < \varepsilon, \eta \leq \pi} \left| \int_{\varepsilon}^{\pi} \int_{\eta}^{\pi} \frac{f(x + s, y + t) - f(x - s, y + t) - f(x + s, y - t) + f(x - s, y - t)}{\operatorname{tg} s/2 \operatorname{tg} t/2} dt ds \right|,$$

$$\varphi_5(x, y) = \sup_{0 < \varepsilon \leq \pi} \left| \int_{\varepsilon}^{\pi} \frac{f(x + s, y) - f(x - s, y)}{\operatorname{tg} s/2} ds \right|,$$

$$\varphi_6(x, y) = \sup_{0 < \eta \leq \pi} \left| \int_{\eta}^{\pi} \frac{f(x, y+t) - f(x, y-t)}{\operatorname{tg} t/2} dt \right|.$$

Further, if $s_{ij}(x, y)$ ($i, j = 0, 1, \dots$) denote the partial sums of the series $\sigma[f]$, then the expressions

$$\sigma_n^{\alpha}(x, y) = \sum_{k=0}^n A_{n-k}^{\alpha-1} s_{kk}(x, y), \quad \alpha > 0,$$

will be called the (C, α) -means of the series $\sigma[f]$.

In the present note we give assertions concerning questions of the existence of the function $\bar{f}_3(x, y)$ and the summability of the functions $\varphi_i(x, y)$ ($i = 0, \dots, 6$); we also study the behavior of the (C, α) -means of the series $\sigma[f]$.

As A. Zygmund showed ⁽¹⁾, if $f(x, y) \in L \log^+ L$, then the function $\bar{f}_3(x, y)$ exists almost everywhere. In ⁽²⁾ he also posed the question: if $f(x, y) \in$

$L(R)$, then does there exist almost everywhere the function $f_3(x, y)$, where

$$\bar{f}_3(x, y) = \lim_{(\varepsilon, \eta)_{\lambda} \rightarrow 0} \int_{\varepsilon}^{\pi} \int_{\eta}^{\pi} \frac{f(x+s, y+t) - f(x-s, y+t) - f(x+s, y-t) + f(x-s, y-t)}{\operatorname{tg} s/2 + \operatorname{tg} t/2} dt ds, \quad (1)$$

where the symbol $(\varepsilon, \eta)_{\lambda} \rightarrow 0$ means that $\varepsilon \rightarrow 0$, $\eta \rightarrow 0$, and $1/\lambda \leq \varepsilon/\eta \leq \lambda$, $\lambda \geq 1$. E. Stein (see ⁽³⁾, Theorem 5) showed that, in contrast to the case of a function of one variable, the limit (1) may fail to exist on a set of positive planar measure even in the case when $\lambda = 1$.

Below we shall show that the scheme of the proof (see ⁽³⁾, Theorem 5) given by E. Stein contains an error. To verify this, we shall need the following

Lemma. Let $(x, y) \in [0, 1/3; 0, 1/3] \equiv I$ and $\varepsilon > 0$. Then, if

$$E^{\varepsilon}(\alpha) \equiv E(\alpha) = \left\{ (x, y) \in I : \frac{1}{xy |\ln x|^{\varepsilon} |\ln y|^{\varepsilon}} \geq \alpha \right\},$$

then for every $\alpha \geq \alpha_0(\varepsilon) > 0$ the measure

$$\operatorname{mes} E(\alpha) \ll \begin{cases} \frac{A}{\alpha} (\ln \alpha)^{1-2\varepsilon}, & \text{if } 1-2\varepsilon > 0, \\ \frac{A'}{\alpha}, & \text{if } 1-2\varepsilon < 0, \end{cases}$$

where A and A' are certain positive constants.

Proof. Put

$$E_1(\alpha) = \left\{ (x, y) \in I : \frac{1}{xy |\ln x|^\varepsilon |\ln y|^\varepsilon} \geq \alpha, y \leq x \right\},$$

$$E_2(\alpha) = \left\{ (x, y) \in I : \frac{1}{xy |\ln x|^\varepsilon |\ln y|^\varepsilon} \geq \alpha, y > x \right\},$$

$$E_3(\alpha) = \left\{ (x, y) \in I : \frac{1}{xy |\ln x|^{2\varepsilon}} \geq \alpha \right\},$$

$$E_4(\alpha) = \left\{ (x, y) \in I : \frac{1}{xy |\ln y|^{2\varepsilon}} \geq \alpha \right\}.$$

It is not difficult to see that $E(\alpha) \subset E_1(\alpha) + E_2(\alpha) \subset E_3(\alpha) + E_4(\alpha)$. Consequently, the measure

$$\text{mes } E(\alpha) \leq \text{mes } E_3(\alpha) + \text{mes } E_4(\alpha). \quad (2)$$

But the set

$$E_3(\alpha) = \left\{ (x, y) \in I : y \leq \frac{1}{\alpha x |\ln x|^{2\varepsilon}} \right\}. \quad (3)$$

Let the number $x_0 \equiv x_0(\alpha, \varepsilon)$ be chosen so that $1/\alpha x_0 |\ln x_0|^{2\varepsilon} = 1/3$. It is clear that

$$1/\alpha^2 < x_0 < 1/\alpha, \quad \alpha \geq \alpha_0(\varepsilon). \quad (4)$$

Thus, taking into account (3) and (4), we shall have

$$\text{mes } E_3(\alpha) \ll \frac{1}{\alpha} \int_{1/\alpha^2}^{1/3} \frac{dx}{x |\ln x|^{2\varepsilon}} + \frac{1}{\alpha} \ll \begin{cases} \frac{A}{2\alpha} (\ln \alpha)^{1-2\varepsilon}, & 1-2\varepsilon > 0, \\ A'/2\alpha, & 1-2\varepsilon < 0; \alpha \geq \alpha_0(\varepsilon). \end{cases} \quad (5)$$

Similarly,

$$\text{mes } E_4(\alpha) \ll \begin{cases} \frac{A}{2\alpha} (\ln \alpha)^{1-2\varepsilon}, & 1-2\varepsilon > 0, \\ A'/2\alpha, & 1-2\varepsilon < 0, \alpha \geq \alpha_0(\varepsilon). \end{cases} \quad (6)$$

From (2), (5), and (6) the validity of the lemma follows.

Analyzing the proof of the lemma, it is not difficult to see that if $1 - 2\varepsilon = 0$, i.e. $\varepsilon = 1/2$, then

$$\text{mes } E(\alpha) \ll \frac{A''}{\alpha} \ln \ln \alpha, \quad \alpha \geq \alpha_0(\varepsilon).$$

Let us note here also that the following estimate is valid:

$$\text{mes } E(\alpha) \geq \frac{B}{\alpha} (\ln \alpha)^{1-2\varepsilon}, \quad 1 - 2\varepsilon > 0, \quad \alpha \geq \alpha_0(\varepsilon), \quad (7)$$

where B is a positive constant. But the proof of relation (7) is more complicated than the proof of the lemma, and therefore we do not give it here.

The assertion that

$$\text{mes } E(\alpha) \geq B'/\alpha, \quad 1 - 2\varepsilon < 0, \quad \alpha \geq \alpha_0(\varepsilon), \quad B' > 0,$$

is, generally speaking, false. Thus, according to the lemma and (7), we shall have

$$\text{mes } E(\alpha) \sim (\ln \alpha)^{1-2\varepsilon}/\alpha, \quad 1 - 2\varepsilon > 0 \quad (\alpha \rightarrow \infty).$$

Now, analyzing the proof of E. Stein (see ³, Theorem 5, pp. 159-160), one can conclude that the function $f_\delta(x, y)$ belongs to the class $L(\log L)^{1-\varepsilon}$ ($\varepsilon > 0$, arbitrarily small) only when $\delta > 1 - \varepsilon > 1/2$; that is, $1 - 2\delta < 0$, and, according to the lemma,

$$\text{mes } E^\delta(\alpha) \leq A'/\alpha, \quad \alpha \geq \alpha_0(\delta),$$

which contradicts assertion (13.6) of E. Stein (see ³, p. 160). Consequently, E. Stein (see ³, p. 154) obtains no contradiction, and thereby he has not proved that if $f(x, y) \in L(\log L)^\alpha$ for some $\alpha \in [0, 1)$, then, generally speaking, the double conjugate series $\bar{\sigma}[f; x, y]$ is not summable almost everywhere by the Poisson method. Consequently, Stein's conclusion on the nonexistence of $f_3(x, y)$, made in the conclusion and referring to Theorem 5, is insufficiently convincing.

We now give results connected with questions of summability of the functions $\varphi_i(x, y)$ ($i = 0, \dots, 6$).

Theorem 1. If $f(x, y) \in L \log L$, then

$$\varphi_i(x, y) \in L^q(R) \quad \text{for all } q \in (0, 1) \quad (i = 0, \dots, 4). \quad (8)$$

If, however, $f(x, y) \in L(\log L)^2$, then

$$\varphi_i(x, y) \in L(R) \quad (i = 0, \dots, 4). \quad (9)$$

Let us note that for functions $f(x, y) \in L(\log L)^\alpha$ for all $\alpha \in [0, 1)$, assertion (8) is, generally speaking, false; and for the validity of assertion (9) the condition $f(x, y) \in L(\log L)^2$ is also essential, since the following theorem is true.

Theorem 2. There exists a nonnegative 2π -periodic function $f_0(x, y) \in L(\log L)^{2-\varepsilon}$ for all $\varepsilon \in (0, 2]$, however $\varphi_i(x, y) \notin L(R)$ ($i = 0, \dots, 4$).

For the summability of the functions $\varphi_i(x, y)$ ($i = 5, 6$), the condition $f(x, y) \in L \log L$ is sufficient, which follows from a known theorem of A. Zygmund⁴.

Theorem 3. Let $f(x, y) \in L(R)$. If $\bar{f}_i(x, y)$ ($i = 1, 2$) are summable, then the series $\bar{\sigma}[f; x]$ and $\bar{\sigma}[f; y]$ are Fourier series respectively of the functions $\bar{f}_1(x, y)$ and $\bar{f}_2(x, y)$; but if $f(x, y) \in L \log L$ and $\bar{f}_3(x, y) \in L(R)$, then the series $\bar{\sigma}[f; x, y]$ is also the Fourier series of the function $\bar{f}_3(x, y)$.

J. Marcinkiewicz⁵ proved that if $f(x, y) \in L \log L$, then almost everywhere $\sigma_n^1(x, y) \rightarrow f(x, y)$ as $n \rightarrow \infty$. In fact, the following is true.

Theorem 4. If $f(x, y) \in L(R)$, then for every $\alpha > 0$, almost everywhere

$$\lim_{n \rightarrow \infty} \sigma_n^\alpha(x, y) = f(x, y).$$

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