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1967

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Abstract

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UDC 517.54

MATHEMATICS

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ON THE EXTENSION OF THE VARIATIONAL METHOD OF G. M. GOLUZIN–P. P. KUFAREV TO MULTIPLY CONNECTED DOMAINS

(Presented by Academician M. A. Lavrent'ev, 9 XI 1966)

The aim of the present work is to extend the well-known variational theorem of G. M. Goluzin ⁽¹⁾ to finitely connected domains. Earlier P. P. Kufarev, jointly with N. V. Genina (Semukhina) ⁽²⁾ (see also ⁽⁵⁾), using a new integral representation of the first variation given by him ⁽³⁾, extended G. M. Goluzin's theorem to families of univalent functions in an annulus. Important results in the same direction were obtained, on the basis of other considerations, by M. Schiffer ⁽⁶⁾ and S. A. Gel'fer ⁽⁷⁾. From the main variational theorem (Theorem 4) of our work there follows the above-mentioned theorem of P. P. Kufarev. The formula indicated in Theorem 1, which generalizes Schwarz' s formula to the case of finitely connected circular domains, is a development and strengthening of the results of V. A. Zmorovich ⁽⁸⁾ and G. Meshkovskii ⁽⁹⁾. Starting from this formula and using a number of considerations of P. P. Kufarev ⁽³⁾, we obtain theorems on differentiability with respect to the parameter t of families of functions $F(w, t)$ and $\Phi(z, t)$, regular in a finitely connected domain, a variational theorem, a generalized Löwner equation, and also a variational formula for a finitely connected domain of the type of the variational formula of M. A. Lavrent'ev for the disk.

Let the real numbers $R_k > 0$ and the complex numbers a_k ($a_0 = a_n = 0$), $k = 0, \dots, n$, be such that the inequalities $|a_i| < R_0 - R_i$, $|a_i - a_j| > R_i + R_j$, $i, j = 1, \dots, n$, are satisfied.

Theorem 1. Let $f(z)$ be a function regular and single-valued inside the intersection of the disk $|z| < R_0$ with the exteriors of the disks $|z - a_k| \leq R_k$, $k = 1, \dots, n$, whose real part on the boundary components $C_k : |\zeta - a_k| = R_k$, $k = 0, \dots, n$, assumes the values $f_k(\zeta)$. Then the formula holds

$$f(z) = \sum_{k=0}^n \frac{\delta_k}{2\pi i} \int_{C_k} f_k(\zeta) H_k(\zeta, z) \frac{d\zeta}{\zeta}$$

$$-\frac{1}{4\pi i} \int_{C_0} f_0(\zeta) \frac{d\zeta}{\zeta} - \frac{1}{4\pi i} \int_{C_n} f_n(\zeta) \frac{d\zeta}{\zeta} + iD,$$

where D is a real number, $\delta_0 = 1$, $\delta_k = -1$, $k = 1, \dots, n$,

$$H_k(\zeta, z) = \frac{\zeta + z}{\zeta - z} + \sum_{\nu=1}^{\infty} \left[\sum_{k_1}^{2\nu} p(\zeta, z; k_1, k_2, \dots, k_{2\nu}) + \sum_{k_1}^{2\nu-1} p(\zeta, z; k, k_1, \dots, k_{2\nu-1}) \right],$$

and

$$p(\zeta, z; k_1, k_2, \dots, k_{2\nu}) =$$

$$= \left[\frac{A_{k_{2\nu}} z}{R_{k_{2\nu}}^2 t_{2\nu-1} + a_{k_{2\nu}} t_{2\nu}} + \frac{A_{h-k_{2\nu}}}{t_{2\nu}} \right] \frac{\zeta}{R_{k_{2\nu}}^2 t_{2\nu-1} - (z - a_{k_{2\nu}}) t_{2\nu}} \prod_{i=1}^{2\nu} R_{k_i}^2,$$

$$A_k = 1 + \frac{1}{2}(\delta_k - \delta_{n-k}), \quad \sum_{k_1}^{k_{2\nu}} = \sum_{\substack{k_1=0 \\ k_1 \neq k}}^n \sum_{\substack{k_2=0 \\ k_2 \neq k_1}}^n \dots \sum_{\substack{k_{2\nu}=0 \\ k_{2\nu} \neq k_{2\nu-1}}}^n.$$

The functions t_p are determined by the recurrence relations

$$t_p = -b_{p-1} t_{p-1} + R_{k_{p-1}}^2 t_{p-2}, \quad t_0 = 1, \quad t_1 = \zeta - a_{k_1},$$

where $b_p = a_{k_{p+1}} - a_{k_p}$, if p is even, and $b_p = \bar{a}_{k_{p+1}} - \bar{a}_{k_p}$, if p is odd.

The functions $f_k(\zeta)$ satisfy the conditions

$$\frac{1}{2\pi i} \int_{C'_m} f_m(\zeta) \frac{d\zeta}{\zeta - a_m} = \sum_{\substack{k=0 \\ k \neq m}}^n \frac{\delta_k}{2\pi i} \int_{C'_k} f_k(\zeta) S_k(\zeta) \frac{d\zeta}{\zeta}, \quad m = 1, \dots, n,$$

where

$$S_k(\zeta) = T_k[\zeta, \bar{L}_m(\infty)] + \frac{\delta_k - 1}{2} T_k(\zeta, \infty),$$

$$T_k(\zeta, z) = \frac{1}{2} H_k(\zeta, z) - \frac{\zeta L'_k(\zeta)}{2 L_k(\zeta)} \bar{H}_k[L_k(\zeta), \bar{z}], \quad L_k(\zeta) = \bar{a}_k + \frac{R_k^2}{\zeta - a_k}.$$

Theorem 2. Let a family of finitely $(n+1)$ -connected domains $G(w, t)$, $a \leq t \leq b$, be given in the w -plane, with the following properties:

- 1) the boundary components $C_k(t)$, $k = 0, \dots, n$, of the domain $G(w, t)$ are simple closed Jordan curves $w = \Omega_k(\lambda, t)$, where $0 \leq \lambda \leq 2\pi$, $k = 0, \dots, n$, and none of the $C_k(t)$, $k = 1, \dots, n$, lies inside another, while all of them lie inside $C_0(t)$.
- 2) the functions $\Omega_k(\lambda, t)$, $k = 0, \dots, n$, are differentiable in t , uniformly with respect to λ , at $t = t_0$, $t_0 \in [a, b]$.
- 3) the curves $C_k(t_0) : w = \Omega_k(\lambda, t_0)$, $k = 0, 1, \dots, n$, are analytic.

Further, let $z = F(w, t)$ be a function mapping conformally $G(w, t)$ onto the intersection of the disk $|z| < R_0(t)$ with the exteriors of the disks $|z - a_k| \leq R_k(t)$, $k = 1, \dots, n$, where the functions $R_k(t)$, $k = 0, \dots, n$, are differentiable at the point $t = t_0$; let there exist

$$\frac{\partial}{\partial t} [\arg F(w, t)] \Big|_{t=t_0}, \quad R_k(t_0) = R_k.$$

Then $F(w, t)$ is differentiable in t , uniformly with respect to w inside $G(w, t_0)$, at $t = t_0$, and for the derivative $F'_t(w, t_0)$ the formula holds

$$\begin{aligned} \frac{\partial F(w, t_0)}{\partial t} = -F(w, t_0) \left\{ \frac{1}{4\pi i} \int_{|\zeta|=R_0} L_0(\zeta) \frac{d\zeta}{\zeta} + \frac{1}{4\pi i} \int_{|\zeta|=R_n} L_n(\zeta) \frac{d\zeta}{\zeta} - iD \right. \\ \left. - \sum_{k=0}^n \frac{\delta_k}{2\pi i} \int_{|\zeta-a_k|=R_k} L_k(\zeta) H_k[\zeta, F(w, t_0)] \frac{d\zeta}{\zeta} \right\}, \end{aligned}$$

where

$$L_k(\zeta) = - \frac{\partial}{\partial t} \ln |F[\Omega_k(\lambda, t), t_0] / R_k(t)| \Big|_{t=t_0}.$$

In this case the relations

$$\frac{1}{2\pi i} \int_{|\zeta-a_m|=R_m} L_m(\zeta) \frac{d\zeta}{\zeta - a_m} = \sum_{\substack{k=0 \\ k \neq m}}^n \frac{\delta_k}{2\pi i} \int_{|\zeta-a_k|=R_k} L_k(\zeta) S_k(\zeta) \frac{d\zeta}{\zeta},$$

$$m = 1, \dots, n. \quad (*)$$

Theorem 3. Under the conditions of Theorem 2, the function $w = \Phi(z, t)$, inverse to $F(w, t)$, is differentiable in t , uniformly with respect to z inside the intersection of the disk $|z| < R_0(t)$ with the exteriors of the disks $|z - a_k| \leq R_k(t)$, $k = 1, \dots, n$, at $t = t_0$, and

$$\left. \frac{\partial \Phi(z, t)}{\partial t} \right|_{t=t_0} = z \frac{\partial \Phi(z, t_0)}{\partial z} \left\{ \frac{1}{4\pi i} \int_{|\zeta|=R_0} L_0(\zeta) \frac{d\zeta}{\zeta} + \frac{1}{4\pi i} \int_{|\zeta|=R_n} L_n(\zeta) \frac{d\zeta}{\zeta} - \sum_{k=0}^n \frac{\delta_k}{2\pi i} \int_{|\zeta-a_k|=R_k} L_k(\zeta) H_k(\zeta, z) \frac{d\zeta}{\zeta} - iD \right\}.$$

where

$$L_k(\xi) = \frac{d}{dt} \ln R_k(t_0) - \operatorname{Re} \left[\frac{1}{\xi \Phi'_\zeta(\xi, t)} \frac{\partial}{\partial t} \Omega_k(\lambda, t) \right]_{t=t_0}.$$

In this case the relations (*) are satisfied.

If in Theorems 2 and 3 one sets $n = 0$ and $n = 1$, then the well-known theorems of P. P. Kufarev (²⁻⁴) are obtained.

Theorem 4. Let a function $f(z)$ be holomorphic and univalent in the intersection of the disk $|z| < R_0$ with the exteriors of the disks $|z - a_k| \leq R_k$, $k = 1, \dots, n$. Suppose that the function $f_k(z, t) = f(z) + tg_k(z)$ is holomorphic and univalent for each $t \in [0, T]$ in the annulus $R_k < |z - a_k| \leq R_k + \varepsilon_k$ for small $\varepsilon_k > 0$, $k = 1, \dots, n$, and maps it onto a domain with boundary continua $C_{R_k}(t)$ and $C_k(t)$. Further, let $f_0(z, t) = f(z) + tg_0(z)$, for each $t \in [0, T]$, be holomorphic and univalent in the annulus $R_0 - \varepsilon_0 \leq |z| < R_0$, $\varepsilon_0 > 0$, and map this annulus onto a domain with boundary continua $C_0(t)$ and $C_{R_0}(t)$. We shall assume that each $C_{R_k}(t)$ lies inside $C_k(t)$, $k = 1, \dots, n$, $C_{R_0}(t)$ —outside $C_0(t)$, none of the curves $C_k(t)$, $k = 1, \dots, n$, lies inside another, and, at the same time, all of them lie inside $C_0(t)$. Let the functions $R_k(t)$, $k = 0, \dots, n$, be differentiable at the point $t = 0$, $R_k(0) = R_k$. Then the function $\Phi(z, t)$, conformally and univalently mapping the domain $K(t)$, which is the intersection of the disk $|z| < R_0(t)$ with the exteriors of the disks $|z - a_k| \leq R_k(t)$, $k = 1, \dots, n$, onto the $(n+1)$ -connected domain $G(w, t)$, whose boundary consists of the continua $C_{R_k}(t)$, $k = 0, \dots, n$, is representable inside $K(t)$ in the form

$$\Phi(z, t) = f(z) + tzf'(z)P(z) + o(t),$$

where

$$P(z) = \sum_{k=0}^n \frac{\delta_k}{2\pi i} \lim_{\rho_k \rightarrow R_k} \int_{|\zeta-a_k|=\rho_k} \operatorname{Re} B_k(\zeta) H_k(\zeta, z) \frac{d\zeta}{\zeta} - \frac{d}{dt} \ln \sqrt{R_0(0)R_n(0)} - iD - \frac{1}{4\pi i} \lim_{\rho_0 \rightarrow R_0} \int_{|\zeta|=\rho_0} \operatorname{Re} B_0(\zeta) \frac{d\zeta}{\zeta}$$

$$-\frac{1}{4\pi i} \lim_{\rho_n \rightarrow R_n} \int_{|\zeta|=\rho_n} \operatorname{Re} B_n(\zeta) \frac{d\zeta}{\zeta}, \quad B_k(\zeta) = \frac{g_k(\zeta)}{\zeta f'(\zeta)}.$$

In this case

$$\sum_{\substack{k=0 \\ k \neq m}}^n \beta_k \frac{d}{dt} \ln R_k(0) - \frac{d}{dt} \ln R_m(0) = \sum_{\substack{k=0 \\ k \neq m}}^n \frac{\delta_k}{2\pi i} \lim_{\rho \rightarrow R_k} \int_{|\zeta-a_m|=\rho} \operatorname{Re} B_k(\zeta) S_k(\zeta) \frac{d\zeta}{\zeta}$$

$$-\frac{1}{2\pi i} \lim_{\rho \rightarrow R_m+0} \int_{|\zeta-a_m|=\rho} \frac{\operatorname{Re} B_m(\zeta) d\zeta}{\zeta - a_m}, \quad m = 1, \dots, n,$$

$$\beta_k = \frac{\delta_k}{2\pi i} \lim_{\rho \rightarrow R_k} \int_{|\zeta-a_k|=\rho} S_k(\zeta) \frac{d\zeta}{\zeta}.$$

If in Theorem 4 one sets $n = 1$, then the theorem of P. P. Kufarev and N. V. Genina (Semukhina) ^(2,5) is obtained. If, then, R_1 is allowed to tend to zero, we obtain the theorem of P. P. Kufarev ⁽⁴⁾.

Theorem 5. Let a family of finite $(n+1)$ -connected domains $G(w, t)$, $t \in [\tau_0, 0]$, be such that:

- 1) $w_0 \in G(w, t)$;
- 2) $G(w, t_1) \subset G(w, t_2)$ for $t_1 > t_2$; $t_1, t_2 \in (\tau_0, 0]$;
- 3) the boundary components $C_k(t)$, $k = 0, \dots, n$, of the domain $G(w, t)$ are simple closed Jordan curves $\Omega_k(\theta, t)$, uniformly with respect to θ , $0 \leq \theta \leq 2\pi$, differentiable with respect to t on $(\tau_0, 0]$.

Then the function $w = \Phi(z, t)$, $\Phi(z_0, t) = w_0$, mapping the intersection of the disk $|z| < R_0(t)$ with the exterior of the disks $|z - a_k| \leq R_k(t)$, $k = 1, \dots, n$, onto the domain $G(w, t)$, satisfies at $(t_0, 0)$ the equation

$$\Phi_t(z, t)|_{t=t_0} = z\Phi_z(z, t_0) \left\{ \frac{1}{4\pi} \Psi_0(2\pi, t_0) + \frac{1}{4\pi} \Psi_n(2\pi, t_0) - iD - \sum_{k=0}^n \frac{\delta_k}{2\pi} \int_0^{2\pi} H_k[a_k + R_k(t)e^{i\theta}, z] d\Psi_k(\theta, t_0) \right\},$$

where

$$\Psi_k(\theta, t_0) = - \lim_{\rho_k \rightarrow 1} \int_0^\theta \frac{\partial}{\partial t} \ln \left| \frac{F\{\Phi[a_k + \rho_k R_k(t)e^{i\theta}, t], t_0\}}{R_k(t)} \right|_{t=t_0} d\theta,$$

$$\Psi_m(2\pi, t_0) = \sum_{\substack{k=0 \\ k \neq m}}^n \lim_{\rho_k \rightarrow 1} \delta_k \int_0^{2\pi} B_k[\theta, \rho_k R_k(t_0)] d\Psi_k(\theta, t_0), \quad m = 1, \dots, n,$$

where

$$B_k(\theta, \rho) = -\frac{\rho e^{i\theta}}{a_k + \rho e^{i\theta}} S_k(a_k + \rho e^{i\theta}), \quad F(w, t) \equiv \Phi^{-1}(z, t).$$

Theorem 5 is a direct generalization of the Löwner-type equation to families of finitely connected domains, obtained earlier for a family of simply connected domains by P. P. Kufarev ⁽³⁾, and for doubly connected domains by I. A. Aleksandrov ⁽¹⁰⁾.

Theorem 6. Let the $(n+1)$ -connected domain G have boundary components

$$C_k: \quad w_k(\theta) = a_k + R_k[1 + \sigma_k(\theta)]e^{i\theta}, \quad 0 \leq \theta \leq 2\pi, \quad \text{Im } \sigma_k(\theta) = 0,$$

$k = 0, \dots, n$. We shall assume that $|\sigma_k(\theta)| < \varepsilon$, where $\varepsilon > 0$ and is small, and that C_k is star-shaped with respect to a_k . Let $\Phi(z)$ map conformally and univalently the domain K , formed by the intersection of $|z| < R_0$ with the exterior of the disks $|z - a_k| \leq R_k$, $k = 1, \dots, n$, onto G . Then, up to small terms of order $o(\varepsilon)$ inside K , the function $\Phi(z)$ is represented in the form

$$\Phi(z) \simeq z \left\{ 1 + \sum_{k=0}^n \frac{\delta_k}{2\pi} \int_0^{2\pi} \sigma_k(\theta) H_k(a_k + R_k e^{i\theta}, z) \frac{d\theta}{1 + a_k R_k^{-1} e^{-i\theta}} - \frac{1}{4\pi} \int_0^{2\pi} \sigma_0(\theta) d\theta - \frac{1}{4\pi} \int_0^{2\pi} \sigma_n(\theta) d\theta - \frac{d}{dt} \ln \right.$$

Moreover,

$$\sum_{\substack{k=0 \\ k \neq m}}^n \beta_k \frac{d}{dt} \ln R_k(0) - \frac{d}{dt} \ln R_m(0) =$$

$$= \sum_{\substack{k=0 \\ k \neq m}}^n \frac{\delta_k}{2\pi} \int_0^{2\pi} \sigma_k(\theta) B_k(\theta, R_k) d\theta - \frac{1}{2\pi} \int_0^{2\pi} \sigma_m(\theta) d\theta, \quad m = 1, \dots, n.$$

From Theorem 6, by simple means, one derives the results obtained earlier for $n = 0$ by M. A. Lavrent'ev ⁽¹¹⁾, for $n = 1$ by I. A. Aleksandrov ⁽¹⁰⁾ and G. V. Sirk ⁽¹²⁾.

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Received
29 X 1966

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