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1967

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Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1967. Volume 176, No. 5

UDC 530.12:531.51

PHYSICS

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A HOMOGENEOUS AXISYMMETRIC COSMOLOGICAL MODEL IN THE ULTRARELATIVISTIC CASE

(Presented by Academician L. I. Sedov on 29 XII 1966)

The present paper is a continuation of ⁽¹⁾, in which a homogeneous axisymmetric solution of the Einstein equations and Maxwell equations was considered under the condition that space is filled with a perfect fluid and, in the comoving system, there is a homogeneous magnetic field directed along the axis of symmetry z . In the comoving system, which is at the same time synchronous, the metric has the form ⁽¹⁾

$$-ds^2 = -c^2 d\tau^2 + a^2(\tau) [d\chi^2 + f^2(\chi) d\varphi^2] + b^2(\tau) dz^2. \quad (1)$$

In the closed model $f(\chi) = \sin \chi$, in the open model $f(\chi) = \text{sh } \chi$, in the flat model $f(\chi) = \chi$.

For dustlike matter $a(\tau)$ and $b(\tau)$ are given in ⁽¹⁾. Below we consider the case of the ultrarelativistic equation of state $e = 3p$.

The equation $T_{0;k}^k = 0$ for $e = 3p$ gives ⁽¹⁾ $(2\dot{a}/a) + (\dot{b}/b) = -3\dot{e}/4e$ (the dot denotes differentiation with respect to proper time τ), i.e.

$$e = \frac{3D}{8\pi k/c^4} (a^2 b)^{-4/3}, \quad D = \text{const}. \quad (2)$$

In the case under consideration, the gravitational equations reduce to three equations (⁽¹⁾, formulas (3)–(5)), of which we shall use the following two together with equation (2):

$$\frac{\dot{a}^2}{c^2 a^2} + 2 \frac{\dot{a}\dot{b}}{c^2 ab} \pm \frac{\alpha^2}{a^2} = \frac{8\pi k}{c^4} (e + W) = \frac{3D}{(a^2 b)^{4/3}} + \frac{B_1^2}{a^4}; \quad (3)$$

$$\frac{2\ddot{a}}{c^2 a} + \frac{\dot{a}^2}{c^2 a^2} \pm \frac{\alpha^2}{a^2} = \frac{8\pi k}{c^4}(-p + W) = -\frac{D}{(a^2 b)^{4/3}} + \frac{B_1^2}{a^4}. \quad (4)$$

The closed model corresponds in (3)–(4) to the upper sign and $\alpha^2 = 1$, the open model to the lower sign and $\alpha^2 = 1$; for the flat model $\alpha^2 = 0$.

1. Let us consider the solution for the case when there is no magnetic field ($B_1^2 = 0$). This question was considered earlier in ^(2, 3).

For $B_1^2 = 0$, from (3) and (4) we obtain for a and b the equations

$$3Y \frac{d^2 Y}{da^2} - 6 \left(\frac{dY}{da} \right)^2 \mp 14\alpha^2 \frac{dY}{da} - 8\alpha^4 = 0, \quad Y \equiv \frac{a\dot{a}^2}{c^2}. \quad (5)$$

$$\frac{D}{b^{4/3}} = a^{2/3} \left[\mp \alpha^2 - \frac{dY}{da} \right]. \quad (6)$$

It is convenient to introduce the parameter ξ according to

$$\frac{1}{\xi^2} = \mp \alpha^2 - \frac{d(a\dot{a}^2/c^2)}{da} \equiv \mp \alpha^2 - \frac{dY}{da}. \quad (7)$$

The possibility of introducing the parameter in this way (for physically real conditions, when $e = 3p > 0$ and the constant $D > 0$) follows from formula (6), which then assumes the form

$$b^{4/3} = \frac{D\xi^2}{a^{2/3}}. \quad (8)$$

Equation (5) is transformed with the aid of (7) into the form

$$\frac{d(a\dot{a}^2/c^2)}{(a\dot{a}^2/c^2)} = 3 \frac{(\mp \alpha^2 \xi^2 - 1) d\xi}{\xi(3 \mp \alpha^2 \xi^2)}. \quad (9)$$

From formulas (7) and (9) we obtain

$$da = (a\dot{a}^2/c^2) 3\xi d\xi / (3 + \alpha^2 \xi^2). \quad (10)$$

Integrating (9) and (10), we obtain a and $\dot{a}$ as functions of ξ , after which $\tau(\xi)$ is also determined. The dependence $b(\xi)$ is expressed by (8).

Closed model. Let us first consider the case $\xi^2 = 3$ (ξ then cannot serve as a parameter). From (7), introducing η according to $(2/3^{1/2})c d\tau = a d\eta$, we obtain

$$a = A(1 - \cos \eta), \quad (2/3^{1/2})c(\tau - \tau_0) = A(\eta - \sin \eta), \quad A = \text{const.} \quad (11)$$

The curve (11) is a cycloid. $b^4 = (3D)^3/a^2$. For $a = 0$, $b = \infty$ and $e = 3p = \infty$. Excluding the case just considered, from (9) and (10) we obtain

$$a\dot{a}/c^2 = A(\zeta^2 - 3)^2/\zeta, \quad A = \text{const}; \quad (12)$$

$$a(\zeta)/A = A_1 + 9\zeta - \zeta^3, \quad A_1 = \text{const}; \quad (13)$$

$$c(\tau - \text{const})/A = \pm 3 \int_{\zeta}^{\zeta} [\zeta a(\zeta)/A]^{1/2} d\zeta. \quad (14)$$

The range of variation of ζ is determined by the requirement that the expression under the radical, $\zeta a(\zeta)/A$, not be negative. In this case various cases are possible, depending on the magnitude of the constant A_1 .

If $A_1 > 6 \cdot 3^{1/2}$, then the cubic (13) has one real root $\zeta = \zeta_1 > 0$; $\zeta a(\zeta)/A \geq 0$ for $0 \leq \zeta \leq \zeta_1$. According to (13), (12), (8), and (2) we have: for $\zeta = \zeta_1$, $a = 0$, $\dot{a} = \infty$, $b = \infty$, and $e = 3p = \infty$; for $\zeta = 0$, $a/A = A_1 \neq 0$, $\dot{a} = \infty$, $b = 0$, and $e = \infty$. Near the value $\zeta = 0$, taking $\tau = \tau_0$ at $\zeta = 0$, we have

$$a/A \approx A_1 + 9(4A_1)^{-1/3}[c(\tau - \tau_0)/A]^{2/3}.$$

It is seen that this formula is valid for $\tau > \tau_0$ and for $\tau < \tau_0$ (on both sides of the value $a/A = A_1$, $e = \infty$). Near $\zeta = \zeta_1$, taking $\tau = \tau_1$ at $\zeta = \zeta_1$, we obtain

$$a/A \approx [27(\zeta_1^2 - 3)^2/12\zeta_1]^{1/3}[c(\tau - \tau_0)/A]^{2/3}.$$

This formula is also valid for $\tau > \tau_1$ and for $\tau < \tau_1$ (on both sides of the value $a = 0$, $e = \infty$). In the interval $2\tau_0 - \tau_1 \leq \tau \leq \tau_1$, the solution is given by (14) with $\text{const} = \tau_0$ and with $\zeta = 0$ as the lower limit of integration. To continue the solution through the value $\tau = \tau_1$, in (14) one must take the corresponding additive constant. Thus we obtain a periodic dependence $a(\tau)$; over one period $a(\tau)/A$ increases from the value $a = 0$, $\dot{a} = \infty$ ($b = \infty$, $e = \infty$, $\zeta = \zeta_1 > 3^{1/2}$), passes through a maximum ($\zeta = 3^{1/2}$), and decreases to the value $a/A = A_1$, $\dot{a} = \infty$ ($b = 0$, $e = \infty$), after which it changes in the reverse order down to the value $a = 0$.

For $A_1 = 6 \cdot 3^{1/2}$, from (13) and (14) we obtain, putting $\zeta = 2 \cdot 3^{1/2} \cos^2 \lambda$:

$$\begin{aligned} a/A &= 6 \cdot 3^{1/2} \sin^2 \lambda (2 \cos^2 \lambda + 1)^2, \\ c\tau/A &= (27/4)^{1/2} (12\lambda + 2 \sin^3 2\lambda - 3 \sin 4\lambda). \end{aligned} \quad (15)$$

For $0 < A_1 < 6 \cdot 3^{1/2}$, the cubic (13) has three real roots $\zeta_1 > 0$, $\zeta_2 < 0$, and $\zeta_3 < \zeta_2 < 0$; $\zeta a(\zeta)/A \geq 0$ for $\zeta_3 \leq \zeta \leq \zeta_2$ and for $0 \leq \zeta \leq \zeta_1$, so that two types of solutions are possible. Solutions with $0 \leq \zeta \leq \zeta_1$ are completely analogous to those considered above. The type of solutions for which $\zeta_3 \leq \zeta \leq \zeta_2$ turns out to be different: for $\zeta = \zeta_3$, $a = 0$, $\dot{a} = \infty$, $b = \infty$, and $e = \infty$; for $\zeta = \zeta_2$ likewise $a = 0$, $\dot{a} = \infty$, $b = \infty$, and $e = \infty$. The function b in this case does not vanish. The asymptotic formulas for $a(\tau)$ near $\zeta = \zeta_3$ and $\zeta = \zeta_2$ are analogous to the one given above for $\zeta = \zeta_1$; the dependence $a(\tau)$ is analytically continued on both sides of the value $a = 0$, $e = \infty$ and has a periodic character.

For $A_1 = 0$, evaluating the integral (14), we obtain

$$(a/A)^2 = \zeta^2(9 - \zeta^2)^2, \quad c(\tau - \text{const})/A = \mp(9 - \zeta^2)^{3/2}. \quad (16)$$

For $\zeta = 0$, $a = 0$, $\dot{a} = \infty$, $b = 0$, and $e = \infty$; for $\zeta^2 = 9$, $a = 0$, $\dot{a} = \infty$, $b = \infty$, and $e = \infty$.

Negative values of the constant A_1 do not lead to new types of solutions, since the corresponding formulas reduce to the formulas with positive A_1 by changing the sign of ξ and a/A .

Open model. Integrating (9) and (10), we obtain

$$\frac{a\dot{a}^2}{c^2} = A \frac{(\xi^2 + 3)^2}{\xi}, \quad A = \text{const}; \quad (17)$$

$$a/A = A_1 + 9\xi + \xi^3, \quad A_1 = \text{const}; \quad (18)$$

$$c(\tau - \tau_0)/A = \pm 3 \int_0^\xi [\xi a/A]^{1/2} d\xi. \quad (19)$$

Again the range of variation of ξ is determined by the requirement that $\xi a/A$ be ≥ 0 . The trinomial (18), for all values of the constant A_1 , has one real root $\xi = \xi_1$. For $A_1 > 0$ this root is $\xi_1 < 0$; $\xi a/A \geq 0$ for $\xi \leq \xi_1$ and for $\xi \geq 0$, so that two types of solutions are possible. The first type corresponds to $\xi \geq 0$. At $\xi = 0$ ($\tau = \tau_0$), $a/A = A_1 \neq 0$, $\dot{a} = \infty$, $b = 0$, and $e = \infty$. The asymptotics near the value $\xi = 0$ ($\tau = \tau_0$) has the same form as that given above for the closed model, and is valid for $\tau > \tau_0$ and $\tau < \tau_0$. As ξ grows from 0 (i.e., as $|\tau - \tau_0|$ increases), a/A increases from the value A_1 to ∞ , and b^4 increases from 0 to D^3/A^2 .

The second type of solutions corresponds to $\xi \leq \xi_1$. At $\xi = \xi_1$, $a = 0$, $\dot{a} = \infty$, $b = \infty$, and $e = \infty$. Near $\xi = \xi_1$ ($\tau = \tau_0$),

$$a/A \approx [27(\xi_1^2 + 3)^2/12\xi_1]^{1/3} [c(\tau - \tau_0)/A]^{1/3}.$$

As $|\xi|$ increases from the value $|\xi_1|$ (i.e., as $|\tau - \tau_0|$ increases), a^2 grows from zero, while b^4 decreases, and as $|\xi| \rightarrow \infty$,

$$a^2 \rightarrow \infty, \quad b^4 \rightarrow D^3/A^2.$$

In this case b nowhere vanishes.

For $A_1 = 0$ we obtain

$$a/A = \xi(9 + \xi^2), \quad c(\tau - \tau_0)/A = \pm [(9 + \xi^2)^{3/2} - 27],$$

$$b^4 = (D^3/A^2)\xi^4(9 + \xi^2)^{-2}. \quad (20)$$

At $\xi = 0$, $a = 0$, $\dot{a} = \infty$, $b = 0$, $e = \infty$; as $|\xi| \rightarrow \infty$, $a^2 \rightarrow \infty$, $b^4 \rightarrow D^3/A^2$.

Flat model (3). From equations (9) and (10) we have ($\alpha^2 = 0$)

$$\frac{a\dot{a}^2}{c^2} = \frac{9A}{\xi}, \quad A = \text{const}; \quad \frac{a}{A} = 9\xi + A_1, \quad A_1 = \text{const};$$

$$\frac{c(\tau - \tau_0)}{A} = \pm 3 \int_{\xi}^{\xi} [\xi a/A]^{1/2} d\xi. \quad (21)$$

For $A_1 > 0$ the expression under the radical in the integral is nonnegative for $\xi \geq 0$ and for $\xi \leq -A_1/9$. Again solutions of two types are possible.

For $\xi \geq 0$, introducing λ according to $\xi = (A_1/9) \text{sh}^2 \lambda$, from (21) and (8) we obtain

$$a/A = A_1 \text{ch}^2 \lambda, \quad c(\tau - \tau_0)/A = (A_1^2/144)(\text{sh} 4\lambda - 4\lambda),$$

$$b^4 = (D^3/9^6 A^2)(a/A)^4 \text{th}^{12} \lambda. \quad (22)$$

At $\tau = \tau_0$ the minimum value $a/A = A_1 \neq 0$ is attained; in this case $\dot{a} = \infty$, $b = 0$, $e = \infty$. Near $\tau = \tau_0$,

$$a/A \approx A_1 + 9(4A_1)^{-1/3}[c(\tau - \tau_0)/A]^{2/3}$$

(as in the closed and open models); as $|\tau| \rightarrow \infty$, $a/A \rightarrow \infty$, $b^4 \rightarrow (a/A)^4(D^3/9^6 A^2)$.

For $\xi \leq -A_1/9$, introducing λ according to $\xi = -(A_1/9) \text{ch}^2 \lambda$, we obtain

$$(a/A)^2 = A_1^2 \operatorname{sh}^4 \lambda, \quad c(\tau - \tau_0)/A = (A_1^2/144)(\operatorname{sh} 4\lambda - 4\lambda),$$

$$b^4 = (D^3/9^6 A^2)(a/A)^4 \operatorname{cth}^{12} \lambda. \quad (23)$$

At $\tau = \tau_0$ the minimum value $a^2 = 0$ is attained; in this case $\dot{a} = \infty$, $b = \infty$, $e = \infty$. Near $\tau = \tau_0$,

$$a/A \approx (81/4\xi_1)^{1/3}[c(\tau - \tau_0)/A]^{2/3};$$

as $|\tau| \rightarrow \infty$, $a^2 \rightarrow \infty$, $b^4 \rightarrow (a/A)^4(D^3/9^6 A^2)$.

For $A_1 = 0$, (21) becomes the Friedmann solution

$$(a/A)^4 = (9^6 A^2/D^3)b^4 = 324c^2(\tau - \tau_0)^2/A^2,$$

which as $\tau \rightarrow \tau_0$ ($e \rightarrow \infty$) differs essentially from (22)–(23).

2. In the case where there is a magnetic field, one must consider the system (3)–(4) with constant $B_1^2 \neq 0$. In this case it is not possible to obtain finite expressions.

Let us consider the plane model for $B_1^2 \neq 0$. Introduce the variables

$$\omega = c^2/(a\dot{a})^2, \quad \xi = -(\omega^2/a) da/d\omega^*. \quad (24)$$

The system (3)–(4) ($\alpha^2 = 0$) is then reduced to the form

$$d\xi/d\omega = (\xi/3\omega^2)[\omega(8B_1^4\xi^2 - 14B_1^2\xi + 6) + \xi(8B_1^2\xi - 3)]; \quad (25)$$

$$b^4 = D^3 a^4 (\xi\omega)^3 / [B_1^2 \xi\omega - (\omega - \xi)]^3. \quad (26)$$

The integral curves of equation (25) for $B_1^2 \neq 0$ are shown in Fig. 1. In the sense of ω , one need consider only values $\omega \geq 0$. According to (2), for nonnegative values of $e = 3p$ having physical meaning, the constant D must be > 0 . It follows from (26) that for $D > 0$ the quantity b^4 is negative for values of ω and ξ inside the region bounded by the integral curves $\xi = 0$ and $\xi = \omega/(1 + B_1^2\omega)$, so that the inner part of this region (shaded in Fig. 1) need not be considered. In the remaining part of the half-plane $\omega \geq 0$, equation (26) has singular points $O(\omega = 0, \xi = 0)$ and $A(\omega = 0, \xi = 3/(8B_1^2))$, lying on the ξ -axis, which is an integral curve. The singular points O and A have a structure more complicated than ordinary singular points.

Fig. 1.

Fig. 1.

Figure 1: Fig. 1.

The arrows indicate the direction of increase of a^2 along an integral curve. The point O corresponds to the minimum of a^2 . When moving along an integral curve in the direction of increasing a^2 , one must pass from the lower half-plane to the upper one through an infinitely distant point $\omega = \text{const}$, $\xi = \mp\infty$, corresponding to a finite value of a^2 . All curves arrive at the point A , at which $a^2 = \infty$, $|\tau| = \infty$.

Near the point O (in the region $\xi < 0$) $\xi \approx -k_1^2 \omega^2$, $k_1^2 = \text{const}$, $a \approx \text{const} \cdot \exp(k_1^2 \omega)$; at the point O , $a \neq 0$, $b = 0$, $e = \infty$. It is essential that the minimum value of a^2 attained at $e = \infty$ is not equal to 0, and the energy density of the magnetic field $W = B_1^2(c^4/8\pi k)/a^4$ is finite. The asymptotics near the singularity $e = \infty$ in the presence of a magnetic field is expressed by formulas (22) for $\lambda \rightarrow 0$ **.

Let us note that for $e = 3p$ (in contrast to the case of dustlike matter) $a^2(\tau)$ and $b^2(\tau)$ are symmetric functions of τ (with an appropriate choice of the time origin).

In the closed and open models, in the presence of a magnetic field, a^2 likewise does not go to 0, since for small a in (3) and (4) the term B_1^2/a^4 dominates over $\pm a^2/a^2$. The asymptotics near $e = \infty$ in the closed and open models is also expressed by formulas (22) for $\lambda \rightarrow 0$.

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Received
27 XII 1966

CITED LITERATURE

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*

$\omega \equiv \text{const}$ corresponds to the isotropic Friedmann solution; for $B_1^2 \neq 0$ the system (3)–(4) has no such solution.

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For $B_1^2 = 0$ the pattern of the integral curves

$$\xi = (9A)^2 \omega^2 / [(9A)^2 \omega - 1]$$

differs substantially from Fig. 1. For $(\omega = 1/(9A)^2, \xi = \mp\infty), a^2 = \infty, |\tau| = \infty$. The curves with $\xi < 0$ correspond to (22), the curves with $\xi > 0$ to (23).

Note: Figure translations are in progress. See original paper for figures.

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