

NON-ISOMORPHISM OF BANACH SPACES OF ANALYTIC FUNCTIONS OF ONE AND OF TWO VARIABLES

G. M. Henkin

1967

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196701.92414>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 513.881

MATHEMATICS

G. M. KHENKIN

NON-ISOMORPHISM OF BANACH SPACES OF ANALYTIC FUNCTIONS OF ONE AND OF TWO VARIABLES

(Presented by Academician A. N. Kolmogorov, 20 XII 1966)

I. In analysis, the idea has become widespread that one or another space of functions of a larger number of variables is, in some sense, “more massive” than the corresponding space of functions of a smaller number of variables. However, of the confirmations of this idea one finds in proofs of the non-isomorphism* of certain “natural” spaces of functions of different numbers of variables. Therefore a result obtained by A. A. Milutin in 1951 ⁽¹⁾ was very unexpected; from it it follows that the spaces of all continuous functions on cubes of different dimensions are isomorphic to one another.

One may try to prove the non-isomorphism of functional spaces by finding their so-called linear dimension**. Denote by D^n the polycylindrical domain of the space of n complex variables, given by the inequalities $|z_i| < 1$ ($i = 1, 2, \dots, n$). Denote by $H(D^n)$ the space of all functions analytic in the domain D^n , with the topology of uniform convergence on compact subsets of the domain D^n . A. N. Kolmogorov ⁽²⁾ showed that the spaces $H(D^n)$ and $H(D^m)$ have different linear dimension if $n \neq m$, and therefore are non-isomorphic.

Let I^k be the k -dimensional unit cube in k -dimensional Euclidean space. Denote by $C^s(I^k)$ the space of all functions, s times continuously differentiable, defined on the cube I^k , with norm

$$\|f(x)\|_{C^s(I^k)} = \sum_{i_1 + \dots + i_k \leq s} \sup_{x \in I^k} \left| \frac{\partial^{i_1 + \dots + i_k} f(x)}{\partial x_1^{i_1} \partial x_2^{i_2} \dots \partial x_k^{i_k}} \right|.$$

Denote by $A(D^n)$ the space of all those analytic functions in the polycylinder D^n which can be continuously extended to the closed polycylinder $\overline{D}^n$, with norm

$$\|f(z)\|_{A(D^n)} = \sup_{z \in \overline{D}^n} |f(z)|.$$

From the results of A. L. Pelczyński^(3,4) it follows that, for any natural s, k , and n ,

$$\dim_l C^s(I^k) = \dim_l A(D^n) = \dim_l C(I),$$

where $C(I)$ is the space of continuous functions on the interval I . Thus, in terms of linear dimension these functional spaces, widespread in analysis, cannot be distinguished.

* Two linear topological spaces are customarily called isomorphic if there exists between them a mutually one-to-one, linear and continuous correspondence (linear homeomorphism).

** Linear dimension is denoted by $\dim_l E$, where E is a linear topological space. It is said that $\dim_l E_1 \leq \dim_l E_2$ if E_1 is isomorphic to a closed linear subspace of the space E_2 ; $\dim_l E_1 = \dim_l E_2$ if $\dim_l E_1 \leq \dim_l E_2$ and $\dim_l E_2 \leq \dim_l E_1$.

Let us now define a finer invariant $r(E)$ of the linear topological space E . We shall say that $r(E_1) \leq r(E_2)$ if E_1 is isomorphic to such a closed subspace $\widetilde{E}_1$ of the space E_2 that there exists a linear continuous operator $S : \widetilde{E}_1^* \rightarrow E_2^*$ extending linear functionals $\varphi \in E_1^*$ to linear functionals $S\varphi \in E_2^*$. We shall write that $r(E_1) = r(E_2)$ if $r(E_1) \leq r(E_2)$ and $r(E_2) \leq r(E_1)$. The invariant $r(E)$ will be called the **growth** of the space E .

A. A. Miljutin has proposed the hypothesis that the spaces $C^s(I^n)$ for different n have different growth and therefore are non-isomorphic. So far only a special case of this hypothesis has been proved, namely, it has been shown⁽⁵⁾ that $r(C^s(I)) \neq r(C^s(I^k))$ for $k \geq 2$.

The spaces $A(D^n)$, apparently, also have different growth for different n . In this direction the following has been proved.

Theorem. *Let $n \geq 2$. Then $r(A(D)) \neq r(A(D^n))$, and consequently the spaces $A(D)$ and $A(D^n)$ are non-isomorphic.*

In the present note a scheme of proof of this theorem is given. It is interesting to recall here one result of A. L. Pelczyński⁽⁴⁾, which in terms of growth can be formulated in the following form: $r(A(D)) \neq r(C(X))$ for any compact Hausdorff space X , where $C(X)$ denotes the space of all continuous functions on X .

- II. Let E be a Banach space, and let E_1^* be the unit ball of the conjugate space, endowed with the weak topology. It is known that then E_1^* is a compact Hausdorff space, and an operator $I_E : E \rightarrow C(E_1^*)$ of an isometric embedding of the space E into the space $C(E_1^*)$ is defined by the formula $I_E x = x(x_1^*)$, where $x \in E$, $x_1^* \in E_1^*$, and $x(x_1^*)$ is the value of the functional x_1^* on the element x . Following A. A. Miljutin, we introduce the following characteristic of subspaces M^* of the space E^* .

Denote by $\chi(M^*)$ the lower bound of the norms of the linear extension operators $S : M^* \rightarrow C^*(E_1^*)$ of linear functionals from M^* to linear functionals from $C^*(E_1^*)$. In other words, the lower bound is taken over the norms of such linear operators S for which the operator $I_E^* S$ is the identity on the subspace M^* . Here I_E^* denotes the operator conjugate to the operator I_E . In the case when no extension S of bounded norm exists, we shall write that $\chi(M^*) = \infty$.

Lemma 1 (A. A. Miljutin). *Let $J : E \rightarrow C$ be some isomorphic embedding of the space E into the space of continuous functions C on a certain compact Hausdorff space. Let $M^* \subset E^*$. Denote by $\chi_J(M^*)$ the lower bound of the norms of the linear operators $S : M^* \rightarrow C^*$ such that the operator $J^* S$ is the identity operator on the subspace M^* . Then*

$$\frac{1}{\|J\|} \chi(M^*) \leq \chi_J(M^*) \leq \|J^{-1}\| \chi(M^*).$$

Here $\|J\|$ denotes the norm of the operator J , and $\|J^{-1}\|$ the norm of the operator inverse to J and defined on the image of the space E under the mapping J .

Let $I_A : A(D) \rightarrow C(\partial D)$ be the operator of the natural isometric embedding of the space $A(D)$ into the space of continuous functions on the boundary ∂D of the disk D . The operator I_A assigns to each function $f(z) \in A(D)$ the continuous function formed by the boundary values of the function $f(z)$ on the circle ∂D . We decompose the subspace $C^*(\partial D)$ into the direct sum of subspaces $C^*(\partial D) = L_1 \oplus M_1$, where L_1 is the subspace of functionals given by absolutely continuous measures on ∂D , and M_1 is the subspace of functionals given by singular and discrete measures on ∂D .

Lemma 2. *The space $A^*(D)$ decomposes into the direct sum of subspaces $A^*(D) = L \oplus M$, where $L = I_A^* L_1$, $M = I_A^* M_1$, and $\chi(M) = 1$, while L is separable.*

Let $f(z) \in A(D)$. Then

$$f(z) = \sum_{k=0}^{\infty} a_k(f) z^k, \quad \text{if } |z| < 1.$$

The coefficients $\{a_k(f)\}$ are linear continuous functionals, i.e. $a_k \in A^*(D)$ for all k . Since

$$a_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-ik\theta} d\theta,$$

we have $a_k(f) \in L$, where $L \subset A^*(D)$ is the subspace of functionals defined in Lemma 2.

Consider the subspace $L_n \subset L$ spanned by the functionals $a_0(f), a_1(f), \dots, a_{n-1}(f)$. Define a linear extension operator $S_n : L_n \rightarrow C^*(\partial D)$ from functionals in L_n to functionals in $C^*(\partial D)$ by the formula

$$S_n \left(\sum_{k=0}^{n-1} \lambda_k a_k \right) (f) = \sum_{k=0}^{n-1} \lambda_k \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-ik\theta} d\theta, \quad \text{where } f \in C(\partial D).$$

It turns out that the extension operator S is almost the best possible among all operators $S : L_n \rightarrow C^*(\partial D)$ such that the operator $I_A^* S$ is the identity on L_n ; namely, the following holds.

Lemma 3. $\|S_n\| \leq \chi(L_n) + 2$.

Lemma 4. There exist two absolute positive constants γ_1 and γ_2 such that

$$\gamma_2 \ln n \leq \|S_n\| \leq \gamma_1 \ln n$$

for every n .

From Lemmas 3 and 4 there follows the important result for us that, for every n ,

$$\chi(L_n) \geq \gamma \ln n,$$

where $\gamma > 0$ is an absolute constant.

We shall regard the space $A(D^2)$ as a subspace of the space $C(\partial D^2)$ of continuous functions on the torus $\partial D \times \partial D$.

Lemma 5. Let N^* be a separable subspace of the space $A^*(D^2)$. Then there exists $t_0 \in \partial D$ such that, for every functional $\varphi \in N^*$, there is an extension preserving the norm to a measure $\mu_\varphi \in C^*(\partial D \times \partial D)$ with support outside the circle $(\partial D \times t_0) \subset (\partial D \times \partial D)$. Let us explain that by the circle $(\partial D \times t_0)$ we mean the set of all points $(z_1, z_2) \in (\partial D \times \partial D)$ such that $z_2 = t_0$. The measure μ_φ has support outside this circle in the sense that its restriction to the circle $(\partial D \times t_0)$ is equal to zero.

Lemma 6. Denote by $M_{t_0}^*$ the subspace of functionals of the space $A^*(D^2)$ that can be represented by measures on the torus $(\partial D \times \partial D)$ with support on the circle $(\partial D \times t_0)$.

Let $T : M_{t_0}^* \rightarrow A^*(D^2)$ be a linear continuous operator which assigns to each $\varphi \in M_{t_0}^*$ some functional $T\varphi \in A^*(D^2)$ such that the functional $T\varphi - \varphi$ can be represented by a measure with support outside the circle $(\partial D \times t_0)$.

Then

$$\|T\varphi\|_{A^*(D^2)} \geq \|\varphi\|_{A^*(D^2)}.$$

Lemma 7. Let M_n^* be an n -dimensional subspace of the space $M_{t_0}^*$, and let T be an operator of the same kind as in Lemma 6. Then

$$\chi(T(M_n^*)) \geq \chi(M_n^*)/\|T\|.$$

Proof of the theorem. Since, for every $n \geq 2$,

$$r(A(D^2)) \leq r(A(D^n)),$$

it suffices to prove that $r(A(D)) \neq r(A(D^2))$. Suppose the contrary, that is, suppose that there exists an isomorphic embedding

$$J : A(D^2) \rightarrow A(D)$$

of the space $A(D^2)$ into $A(D)$, that there exists

linear continuous operator $S : A^*(D^2) \rightarrow A^*(D)$ such that J^*S is the identity operator in the space $A^*(D^2)$. By Lemma 2, $A^*(D) = L \oplus M$, where the subspace L is separable, and the subspace M is such that $\chi(M) = 1$. By Lemma 5, there is a $t_0 \in \partial D$ such that for every functional $\varphi \in J^*L$ there is an extension of it to a measure $\mu_\varphi \in C^*(\partial D \times \partial D)$ with support outside the circle $(\partial D \times t_0) \subset \partial D \times \partial D$. Let $f(z_1, z_2) \in A(D^2)$, where $(z_1, z_2) \in D^2$. For $z_2 = t_0$ and $|z_1| < 1$,

$$f(z_1, t_0) = \sum_{k=0}^{\infty} a_k(f, t_0) z_1^k.$$

Denote by M_n^* the n -dimensional subspace of functionals from $A^*(D^2)$ spanned by the functionals $a_0(f, t_0), a_1(f, t_0), \dots, \dots, a_{n-1}(f, t_0)$. By Lemmas 4 and 5, for every n , $\chi(M_n^*) \geq \gamma \ln n$, where $\gamma > 0$ is an absolute constant. Let $M_{t_0}^*$ be the subspace of functionals defined in Lemma 6. Consider the operator $T : M_{t_0}^* \rightarrow A^*(D^2)$, which to each functional $\varphi \in M_{t_0}^*$ assigns the functional $T\varphi = J^*P_M S\varphi \in A^*(D^2)$, where P_M denotes the projection of the space $A^*(D) = L \oplus M$ onto the direct summand M . Denote by P_L the projection of $A^*(D)$ onto the subspace L .

Then

$$\varphi - T\varphi = J^*(P_M + P_L)S\varphi - J^*P_M S\varphi = J^*P_L S\varphi.$$

Since the functional $J^*P_L S\varphi \in J^*L$, by construction the functional $T\varphi - \varphi$ can be given by a measure with support outside the circle $(\partial D \times t_0)$. Since $M_n^* \subset M_{t_0}^*$, from Lemma 7 we have

$$\chi(T(M_n^*)) \geq \frac{1}{\|T\|} \chi(M_n^*) \geq \frac{\gamma}{\|T\|} \ln n,$$

where n is arbitrary and $\gamma > 0$ is an absolute constant. It follows that $\chi(T(M_{t_0}^*)) = \infty$. We shall show, on the other hand, that $\chi(T(M_{t_0}^*)) < \infty$.

For this, by Lemma 1 it is enough to verify that, for some isomorphic embedding $J_0 : A(D^2) \rightarrow C(\partial A)$, the inequality $\chi_{J_0}(T(M_{t_0}^*)) < \infty$ holds. Let $J_0 = I_{AJ}$, where $I_A : A(D) \rightarrow C(\partial D)$ is the isometric embedding operator defined above. By Lemma 6 the operator $T = J^* P_{MS}$ is invertible, and therefore the operator J^* is also invertible; moreover the operator $(J^*)^{-1}$ is defined on the subspace $T(M_{t_0}^*) \subset A^*(D^2)$. Denote by $S_M : M \rightarrow C^*(\partial D)$ a linear continuous operator extending the functionals from $M \subset A^*(D)$ to linear functionals from $C^*(\partial D)$. This operator exists by Lemma 2.

Since $(J^*)^{-1}T(M_{t_0}^*) \subset M$, the linear continuous operator

$$S_0 = S_M(J^*)^{-1} : T(M_{t_0}^*) \rightarrow C^*(\partial D)$$

is defined, and such that $J_0^* S_0$ is the identity operator on the subspace $T(M_{t_0}^*) \subset A^*(D^2)$. Consequently, $\chi(T(M_{t_0}^*)) < \infty$, and we have obtained the desired contradiction. The theorem is proved.

Moscow State University
named after M. V. Lomonosov

Received
5 XII 1966

REFERENCES

1. A. A. Miljutin, in: *Theory of Functions, Functional Analysis and Their Applications*, vol. 2, Kharkov, 1966.
2. A. N. Kolmogorov, DAN, 120, No. 2, 239 (1958).
3. A. Pełczyński, Vestn. Leningrad. Univ., No. 13, issue 3, 22 (1962).
4. A. Pełczyński, Studia Math., 24, No. 3, 285 (1964).
5. G. M. Khenkin, DAN, 172, No. 1 (1967).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.