

# ELEMENTARY PARTICLES IN THE FIELD OF A PLANE ELECTROMAGNETIC WAVE

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**Abstract**

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**PHYSICS**

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## **ELEMENTARY PARTICLES IN THE FIELD OF A PLANE ELECTROMAGNETIC WAVE**

*(Presented by Academician V. A. Fock on 30 VI 1966)*

The equation for a particle in an electromagnetic field has the form

$$(\hat{\nabla} - i\hat{A} + m)\psi = 0 \quad (1)$$

(the charge is included in  $A$ ). Here  $\nabla = (\nabla_l) = \left(\frac{\partial}{\partial x_l}\right)$ ;  $x_4 = ix_0 = ict$ ;

$A = (A_l)$  is the vector potential of the electromagnetic field;  $\hat{A} = A_l \gamma^l$ ;  $\gamma^l$  are the basic matrices of the equation. Exact solutions of equation (1) have so far been obtained only in a few cases. For particles with a single value of the rest mass (see <sup>(1)</sup>)

$$\hat{q}^n(\hat{q}^2 - q^2) = 0, \quad (2)$$

where  $q$  is an arbitrary (including operator) 4-vector with mutually commuting components. Denote  $H = \hat{\nabla} + m$ ,  $U = -i\hat{A}$ , and let  $\psi = \psi_0 e^{ipx}$  be the wave function of a free particle ( $px = p_l x_l$ ,  $p^2 + m^2 = 0$ ). Then  $H\psi = (i\hat{p} + m)\psi = 0$ , while equation (1) takes the form  $(H + U)\psi_1 = 0$ . Formally the latter equation may be written in the form  $H(1 + H^{-1}U)\psi_1 = 0$ . Comparing with  $H\psi = 0$  (see (2)), we conclude that  $(1 + H^{-1}U)\psi_1 = \psi$ , whence

$$\psi_1 = (1 + H^{-1}U)^{-1}\psi = [1 - H^{-1}U + (H^{-1}U)^2 - \dots] \psi. \quad (3)$$

As shown in <sup>(2)</sup>, the operator inverse to  $H$ , owing to (2), can be represented in the form  $H^{-1} = P_1(\hat{\nabla}) + (m^2 - \square)^{-1}P_2(\hat{\nabla})$ , where  $P_1$  and  $P_2$  are certain polynomials. Obviously,  $(m^2 - \square)^{-1}e^{iqx}$ , and the action of  $(m^2 - \square)^{-1}$  on a function of a more general form is determined from this by expanding the latter in a series or Fourier integral. Let

$$A = af(\varphi), \quad \varphi = kx = k_l x_l, \quad k^2 = ka = 0, \quad (4)$$

i.e. the electromagnetic field has the character of a plane wave. From (3), (4) it is clear that the expression for  $\psi_1$  will have the form

$$\psi_1 = \chi(\varphi)e^{ipx}, \quad \chi(\varphi) = \Phi(k, p, \varphi)\psi_0, \quad (5)$$

where  $\Phi(k, p, \varphi)$  is a matrix whose elements depend on  $x_l$  only through  $\varphi = k_l x_l$ . The preceding arguments, despite their formal character, have the significance that they lead to the conclusion that it is possible to seek a solution of equation (1) for the field (4) in the form (5) in the case of a particle with arbitrary spin. Substituting (4), (5), and (1), we obtain

$$\hat{k}\chi' + (i\hat{b} + m)\chi = 0, \quad \chi' = d\chi/d\varphi, \quad b = p - af. \quad (6)$$

Thus, in the general case, the problem is reduced to solving a system of ordinary first-order differential equations for the function  $\chi = \chi(\varphi)$ .

The matrix  $\hat{k}$  is singular, since according to (2), (4)  $\hat{k}^{n+2} = k^2 \hat{k}^n = 0$ . Therefore the differential equation (6) contains additional algebraic conditions on the function  $\chi$ . Their extraction is complicated by the fact that the matrix  $\hat{k}$  is not reducible to diagonal form, since its minimal equation  $\hat{k}^{n+2} = 0$  has multiple zero roots. This makes it difficult to apply the method of projective operators <sup>(4)</sup>. Therefore we multiply equation (6) by the matrices of the invariant bilinear form  $\eta$ . We obtain the equation

$$\alpha\chi' + \eta(i\hat{b} + m)\chi = 0, \quad (\alpha = \eta\hat{k} = -\alpha^+), \quad (7)$$

which is completely equivalent to equation (6), since  $\eta^2 = 1$ . However, in (7) the matrix  $\alpha$  will be anti-Hermitian <sup>(4)</sup>, as a result of which its minimal equation will not contain multiple roots and it can be written in the form  $aP_0(a) = 0$ , with  $P_0(0) \neq 0$ . Therefore  $P_0(a)/P_0(0)$  will be a projective operator selecting, without exception, all vectors  $\chi$  for which  $a\chi = 0$  <sup>(1)</sup>. Multiplying (7) by  $P_0(a)$ , we obtain  $P_0(a)\eta(i\hat{b} + m)\chi = 0$ , whence it follows that  $\eta(i\hat{b} + m)\chi = \alpha\chi_1 = \eta\hat{k}\chi_1$ , where  $\chi_1$  is an arbitrary vector. Hence, in turn, it follows that

$$\chi = (i\hat{b} + m)^{-1}\hat{k}\chi_1. \quad (8)$$

The operator  $(i\hat{b} + m)^{-1}$  is not difficult to find with the aid of the minimal equation (2); it has the form (see <sup>(2)</sup>)

$$(i\hat{b} + m)^{-1} = \frac{1}{m} \sum_{k=0}^{n-1} \left( -\frac{i}{m} \hat{b} \right)^k + \left( -\frac{i}{m} \right)^n \frac{\hat{b}^n(m - i\hat{b})}{b^2 + m^2}, \quad (9)$$

The function  $\chi_1$  in (8), generally speaking, may depend on  $\varphi$  in an arbitrary manner. The simpler case is that in which the dependence on  $\varphi$  is concentrated

in a scalar factor  $F(\varphi)$  with a constant vector  $\chi_0$ . Taking (9) into account, in order to get rid of the denominator  $b^2 + m^2 = a^2 f^2 - 2apf$ , which depends on  $\varphi$ , it is convenient to include explicitly in  $F$  the factor  $b^2 + m^2$  and to seek the solution in the form

$$\chi = F(\varphi)(b^2 + m^2)(i\hat{b} + m)^{-1}\hat{k}\chi_0. \quad (10)$$

We shall apply the general approach described above to particles with the lowest spins  $1/2, 0$ , and  $1$ .

1. **Spin  $1/2$ .** In this case the Dirac algebra holds:

$$\hat{p}\hat{q} + \hat{q}\hat{p} = 2pq, \quad \hat{q}^2 = q^2, \quad (11)$$

$$\hat{k}^2 = \hat{k}\hat{a} + \hat{a}\hat{k} = \hat{k}\hat{a}\hat{k} = 0, \quad (b^2 + m^2)(i\hat{b} + m)^{-1} = m - i\hat{b}.$$

Substituting  $\chi = F(m - i\hat{b})\hat{k}\chi_0$  into (6), we obtain

$$\left(F' - \frac{b^2 + m^2}{2ikp}F\right)\hat{k}\chi_0 = 0, \quad (12)$$

whence it follows that

$$F = \exp\left(-\frac{i}{2kp} \int (b^2 + m^2) d\varphi\right). \quad (13)$$

Thus, the solution of equation (1) has the form (see (5))

$$\psi = (m - i\hat{b})\hat{k}\chi_0 \exp i \left[ px - \frac{1}{2kp} \int (b^2 + m^2) d\varphi \right], \quad (14)$$

where  $\chi_0$  is an arbitrary bispinor. This solution was obtained in another way and in another form by D. M. Volkov <sup>(5)</sup>.

## 2. Spins 0 and 1

In this case the Duffin-Kemmer algebra is valid

$$\hat{p}\hat{q}\hat{r} = \hat{r}\hat{p}\hat{q} = pq \cdot r + rp \cdot q, \quad \hat{q}^3 = q^2\hat{q}, \quad \hat{q}\hat{r}\hat{q} = qr \cdot \hat{q}, \quad (15)$$

$$\hat{k}^3 = \hat{k}\hat{a}\hat{k} = \hat{a}\hat{k}\hat{a} = \hat{k}^2\hat{a} + \hat{a}\hat{k}^2 = 0,$$

$$(b^2 + m^2)(\hat{b} + m)^{-1} = \frac{1}{m} [b^2 + m^2 + i\hat{b}(\hat{b} - m)]. \quad (16)$$

Substituting (10), (16) into (6), with the same value of  $F$  (13), after some transformations, taking (15) into account, we obtain

$$\{F'(\hat{p} + m)\hat{k}^2(\hat{p} + m) + (Ff)'[\hat{p}, \hat{k}^2\hat{a}] + (Ff^2)' \hat{k}^2\hat{a}^2\} \chi_0 = 0. \quad (17)$$

In order that this relation be satisfied for constant  $\chi_0$ , it is necessary that

$$(i\hat{p} + m)\chi_0 = 0, \quad \hat{k}^2\hat{a}^2\chi_0 = 0, \quad [\hat{p}, \hat{k}^2\hat{a}]\chi_0 = 0. \quad (18)$$

The conditions  $\hat{k}^2\hat{a}^2\chi_0 = 0$  and  $\hat{k}^2\hat{a}\chi_0 = 0$  follow one from the other as a result of multiplication by  $\hat{a}$ . Therefore the third equation (18) is a consequence of the first two. We shall now consider separately the cases of spin 0 and 1.

- a) **Spin 0.** In this case  $\hat{k}^2\hat{a} = 0$ , and only the condition is imposed on the function  $\chi_0$  that it be a solution of the equation for a free particle,

$$(i\hat{p} + m)\chi_0 = 0.$$

- b) **Spin 1.** In this case the wave function has the form  $\psi = (\psi_k, \psi_{ln})$ , where  $\psi_k$  is a 4-vector and  $\psi_{ln}$  is an antisymmetric tensor. The operator  $\hat{p}$  acts on  $\psi$  in the following way:

$$\hat{p}\psi = \hat{p}(\psi_k, \psi_{ln}) = (p_l\psi_{kl}, p_n\psi_l - p_l\psi_n). \quad (19)$$

The first of equations (18) leads to the relations  $p_k\psi_k = 0$ ,  $\psi_{ln} = \frac{i}{m}(p_l\psi_n - p_n\psi_l)$ , while from the second it follows that  $k_n\psi_n = 0$  and  $a_n k_l \psi_{nl} = -k p \cdot a_n \psi_n = 0$ . Thus, the conditions (18) are satisfied if

$$p\psi_l = k\psi_l = 0, \quad \psi_l = C\varepsilon_{lmnr}p_m k_n a_r, \quad (20)$$

where  $\varepsilon_{lmnr}$  is the Levi-Civita symbol.

Thus, the exact solution of the problem of a particle in the field of a plane electromagnetic wave can be written in a single general form at once for three different particles with spins 1/2, 0, 1,

$$\psi = (b^2 + m^2)(\hat{b} + m)^{-1} \hat{f} \chi_0 \exp i \left( px - \int \frac{b^2 + m^2}{2kp} d\varphi \right), \quad (21)$$

where the function  $\chi_0$ , independent of  $\varphi$ , is arbitrary in the case of spin 1/2, for spin 0 is subject to the condition  $(i\hat{p} + m)\chi_0 = 0$ , and for spin 1 has the form (20).

Let us note that this solution is valid not only in the case of a monochromatic electromagnetic wave, but also in the case of an arbitrary function  $f(\varphi)$ . Obviously, the method set forth here is in principle applicable also to particles with higher spins, although the absence or complexity of the algebra of the matrices  $\gamma_\nu^l$ , naturally, will make the calculations difficult.

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*Note: Figure translations are in progress. See original paper for figures.*

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