

VIBRATIONS OF THREE-LAYER BEAMS

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Abstract

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THEORY OF ELASTICITY

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VIBRATIONS OF THREE-LAYER BEAMS

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§ 1. The problem of flexural vibrations of a three-layer beam, under certain assumptions, reduces ^(1,2) to the solution of a single partial differential equation for the displacement function $\chi(x, t)$

$$(1 - \vartheta k \nabla^2) \nabla^2 \nabla^2 \chi(x, t) + \frac{\Omega l^4}{D} \frac{\partial^2}{\partial t^2} (1 - k \nabla^2) \chi(x, t) = 0, \quad (1,1)$$

where $\nabla^2 = \partial^2 / \partial x^2$; ϑ, k, Ω, D are constants depending on the elastic, geometric, and mass characteristics of the three-layer bar ⁽¹⁾; $\chi(x, t)$ is a dimensionless displacement function of the longitudinal coordinate x and time t , related to the deflection $w(x, t)$ by the formula

$$w(x, t) = l(1 - k \nabla^2) \chi(x, t) \quad (1,2)$$

(l is some linear quantity, for example the length of the bar).

We shall seek a solution of the linear differential equation (1,1) in the form

$$\chi(x, t) = \chi(x) e^{i\omega t}, \quad (1,3)$$

where the amplitude $\chi(x)$, in the general case, is a complex function of the real variable x ; ω is the vibration frequency. Substituting (1,3) into (1,1), we find

$$(1 - \vartheta k \nabla^2) \nabla^2 \nabla^2 \chi(x) - \omega_*^2 (1 - k \nabla^2) \chi(x) = 0, \quad (1,4)$$

where $\nabla^2 = \frac{d^2}{dx^2}$; $\omega_*^2 = \frac{\Omega l^4}{D} \omega^2$ is the dimensionless frequency.

Represent the particular solution of (1,4) in the form

$$\chi(x) = e^{\alpha x}. \quad (1,5)$$

Substituting (1,5) into (1,4), and canceling $e^{\alpha x} \neq 0$, we obtain

$$y^3 + a_4 y^2 + a_2 y + a_0 = 0, \quad (1,6)$$

where $y = \alpha^2$; $a_4 = -1/\vartheta k$; $a_2 = \omega_*^2/\vartheta$; $a_0 = -\omega_*^2/\vartheta k$.

It can be shown that the cubic equation (1,6) has three real roots: one negative ($y_1 = -\lambda^2$) and two positive ($y_2 = \mu^2$, $y_3 = \nu^2$), related by

$$\omega_*^2 = \lambda^4 \frac{1 + \vartheta k \lambda^2}{1 + k \lambda^2}; \quad y_{2,3} = \frac{1 + \vartheta k \lambda^2}{2\vartheta k} \left(1 \mp \sqrt{1 - \frac{4\vartheta k \lambda^2}{(1 + \vartheta k \lambda^2)(1 + k \lambda^2)}} \right). \quad (1,7)$$

For $4\vartheta k/(1 + \vartheta k \lambda^2)(1 + k \lambda^2) \ll 1$, the estimates

$$y_2 \geq \frac{\lambda^2}{1 + k \lambda^2}, \quad y_3 \leq \frac{1}{\vartheta k} + \frac{k \lambda^2}{1 + k \lambda^2}. \quad (1,8)$$

are valid.

The general solution of equation (1,4) can be represented in the form

$$\begin{aligned} \chi(x) = \sum_{i=1}^6 \bar{C}_i e^{\alpha_i x} = C_1 \sin \lambda x + C_2 \cos \lambda x + C_3 \operatorname{sh} \mu x + C_4 \operatorname{ch} \mu x \\ + C_5 \operatorname{sh} \nu x + C_6 \operatorname{ch} \nu x \end{aligned} \quad (1,9)$$

(where $\bar{C}_i, C_i$ are constants).

§ 2. Let us consider a number of boundary-value problems.

A. Hinged edges. The boundary conditions at $x = 0$, $x = 1$ will be

$$\chi = \nabla^2 \chi = \nabla^2 \nabla^2 \chi = 0. \quad (2,1)$$

Substituting (1,9) into (2,1), we arrive at a homogeneous system of linear algebraic equations for the coefficients C_i ($\bar{C}_i$). The condition for a nonzero solution has the form

$$\Delta = \operatorname{sh} a_1 \operatorname{sh} a_3 \operatorname{sh} a_5, \quad (2,2)$$

where Δ is the determinant composed of the coefficients of $\bar{C}_i$ in the indicated system.

Consequently, one of the roots is $a_1 = i n \pi$.

Fig. 1. Variation of the dimensionless vibration frequency ω_* as a function of the shear parameter k for hinged support of the ends without diaphragms

Fig. 1 and Fig. 2

Figure 1: Fig. 1 and Fig. 2

(a) and with absolutely rigid diaphragms at the ends (b); the first five modes; $\vartheta = 0.01$

Fig. 2. Variation of the first mode shape of a three-layer beam as a function of the shear parameter k for hinged ends (a) (diaphragms absent); one end clamped and the other hinged (b)

Substituting a_1 into (1,7), we obtain an explicit expression for the vibration frequency (Fig. 1)

$$\omega_*^2 = n^4 \pi^4 \frac{1 + \vartheta k n^2 \pi^2}{1 + k n^2 \pi^2}. \quad (2,3)$$

To calculate the vibration modes we shall use the first five relations (2,1). We obtain a system of linear algebraic equations for the coefficients C_i . Solving it, we find

$$C_1 = 1, \quad C_i = 0 \quad (i = 2, 3, \dots, 6). \quad (2,4)$$

As a result the dimensionless deflection $w(x)$ is equal to (see Fig. 2a)

$$w(x) = (1 + k n^2 \pi^2) \sin n \pi x. \quad (2,5)$$

B. Hinged edges. At the edges there are diaphragms, absolutely flexible out of their plane and absolutely rigid in shear in their plane (Fig. 1, 3). The boundary conditions for $x = 0, x = 1$ take the form

$$(\nabla^2 \chi)_{,x} = \nabla^2 (1 - \vartheta k \nabla^2) \chi = (1 - k \nabla^2) \chi = 0. \quad (2,6)$$

The determinant Δ in this case will be

$$\begin{aligned} \Delta(\vartheta, k, \omega_*) = & \sum_{i,j,k} (\alpha_k^2 - \alpha_i^2)(\alpha_j^2 - \alpha_i^2) \alpha_j^3 \alpha_k^3 \alpha [1 - \vartheta k(\alpha_k^2 + \alpha_i^2) + \vartheta k^2 \alpha_i^2 \alpha_k^2] \times \\ & \times [1 - \vartheta k(\alpha_j^2 + \alpha_i^2) + \alpha_i^2 \alpha_j^2] \operatorname{sh} \alpha_i (1 - \operatorname{ch} \alpha_j \operatorname{ch} \alpha_k) + \\ & + \frac{1}{2} \operatorname{sh} \alpha_1 \operatorname{sh} \alpha_3 \operatorname{sh} \alpha_5 \sum_{i,j,k} (\alpha_k^2 - \alpha_j^2) \alpha_j^6 [1 - \vartheta k(\alpha_k^2 + \alpha_j^2) + \vartheta k^2 \alpha_j^2 \alpha_k^2] = 0, \end{aligned} \quad (2,7)$$

where the indices i, j, k (1, 3, 5) are permuted cyclically three times in the sums. Joint solution of (1.6) and (2.6) makes it possible to determine the dependence

of the vibration frequency ω_* on the parameters ϑ , k . The results of the calculations are presented in Fig. 1. The vibration modes are calculated by the method indicated above. In this case all the coefficients C_i are different from zero.

Fig. 3. Dependence of the roots α_i of the frequency equation on the shear parameter k for hinged support of the edges with absolutely rigid diaphragms: first six modes; $\vartheta = 0.01$

Fig. 4. Variation of the dimensionless vibration frequency ω_* as a function of the shear parameter k ; both ends of the beam are freely clamped; first eight modes; $\vartheta = 0.01$

C. Free clamping along the edges $x = 0$, $x = 1$ (Fig. 4). The boundary conditions are

$$(1 - k\nabla^2)\chi = \chi_{,xx} = (\nabla^2\chi)_{,xx} = 0. \quad (2.8)$$

The determinant Δ is equal to

$$\begin{aligned} \Delta(\vartheta, k, \omega_*) = & \sum_{i,j,k} \alpha_i^2 \alpha_j \alpha_k (\alpha_k^2 - \alpha_i^2)(\alpha_j^2 - \alpha_i^2) \times \\ & \times [1 - k(\alpha_k^2 + \alpha_j^2) + k^2 \alpha_j^2 \alpha_k^2] \operatorname{sh} \alpha_i (1 - \operatorname{ch} \alpha_j \operatorname{ch} \alpha_k) + \\ & + \frac{1}{2} \operatorname{sh} \alpha_1 \operatorname{sh} \alpha_3 \operatorname{sh} \alpha_5 \sum_{i,j,k} \alpha_i^2 \alpha_k^2 (\alpha_k^2 - \alpha_i^2)^2 (1 - k\alpha_j^2)^2 \\ & (i, j, k \text{ —threefold cyclic permutation}). \end{aligned} \quad (2.9)$$

7. Free clamping along the edge $x = 0$ and hinged support along the edge $x = 1$. The boundary conditions are

$$\chi_{,xx} = (\nabla^2\chi)_{,xx} = (1 - k\nabla^2)\chi = 0 \quad (x = 0); \quad (2.10)$$

$$\chi = \nabla^2\chi = \nabla^2\nabla^2\chi = 0 \quad (x = 1). \quad (2.11)$$

The conditions for determining the vibration frequencies have the form

$$\begin{aligned} \Delta(\vartheta, k, \omega_*) = & \sum_{i,j,k} \alpha_k \alpha_j (\alpha_k^2 - \alpha_j^2)^2 (\alpha_k^2 - \alpha_i^2)(\alpha_j^2 - \alpha_i^2)(1 - k\alpha_i^2) \times \\ & \times \operatorname{sh} \alpha_i \operatorname{ch} \alpha_j \operatorname{ch} \alpha_k = 0 \end{aligned} \quad (2.12)$$

(where the indices i, j, k (1, 3, 5) are cyclically permuted three times). For the coefficients C_i of the vibration modes, compact expressions in terms of the roots α_i are obtained in this case (Fig. 2b).

D. Free clamping along the edge $x = 0$; the edge $x = 1$ is free. The boundary conditions are

$$(1 - k\nabla^2)\chi = \chi_{,xx} = (\nabla^2\chi)_{,xx} = 0 \quad (x = 0); \quad (2.13)$$

$$\nabla^2\chi = \nabla^2\nabla^2\chi = (1 - \vartheta k\nabla^2)\nabla^2\chi_{,xx} = 0 \quad (x = 1). \quad (2.14)$$

The condition for determining the vibration frequencies takes the form

$$\begin{aligned} \Delta(\vartheta, k, \omega_*) &= \sum_{i,j,k} \alpha_i^2(\alpha_i^2 - \alpha_j^2)(\alpha_k^2 - \alpha_i^2)\alpha_i\alpha_j\alpha_k \times \\ &\times [\alpha_j^4(1 - \vartheta k\alpha_j^2)(1 - k\alpha_k^2) + \alpha_k^4(1 - \vartheta k\alpha_k^2)(1 - k\alpha_j^2)] \operatorname{ch} \alpha_i + \\ &+ \sum_{i,j,k} \alpha_i^2(\alpha_i^2 - \alpha_j^2)(\alpha_k^2 - \alpha_i^2)\alpha_i\alpha_j\alpha_k \times \\ &\times [\alpha_j^4(1 - \vartheta k\alpha_j^2)(1 - k\alpha_k^2) + \alpha_k^4(1 - \vartheta k\alpha_k^2)(1 - k\alpha_j^2)] \operatorname{ch} \alpha_i \operatorname{sh} \alpha_j \operatorname{sh} \alpha_k + \\ &+ \alpha_1^3\alpha_3^3\alpha_5^3 \operatorname{ch} \alpha_1 \operatorname{ch} \alpha_3 \operatorname{ch} \alpha_5 \sum_{i,j,k} (\alpha_i^2 - \alpha_j^2)(1 - \vartheta k\alpha_k^2)(1 - k\alpha_k^2) = 0 \end{aligned} \quad (2.15)$$

(the indices i, j, k (1, 3, 5) are cyclically permuted three times).

E. Free clamping along the edge $x = 0$; on the free edge $x = 1$ there is a diaphragm, absolutely flexible out of its plane and absolutely rigid in shear in its plane. The edge conditions will be

$$(1 - k\nabla^2)\chi = \chi_{,xx} = (\nabla^2\chi) = 0 \quad (x = 0); \quad (2.16)$$

$$\nabla^2\chi_{,xx} = \nabla^2(1 - \vartheta k\nabla^2)\chi = \nabla^2\nabla^2\chi_{,xx} = 0 \quad (x = 1). \quad (2.17)$$

The determinant is equal to

$$\Delta(\vartheta, k, \omega_*) = \sum_{i,j,k} \alpha_i^3(\alpha_j^3 - \alpha_i^2)(\alpha_k^2 - \alpha_i^2)\alpha_i\alpha_j\alpha_k \times$$

$$\begin{aligned}
& \times [\alpha_j^4(1 - \vartheta k \alpha_j^2)(1 - k \alpha_k^2) + \alpha_k^4(1 - \vartheta k \alpha_k^2)(1 - k \alpha_j^2)] \operatorname{sh} \alpha_i + \\
& + \sum_{i,j,k} \alpha_i(\alpha_i^2 - \alpha_j^2)(\alpha_k^2 - \alpha_i^2) \alpha_i^3 \alpha_j^3 \alpha_k^3 \times \\
& \times [(1 - \vartheta k \alpha_j^2)(1 - k \alpha_k^2) + (1 - \vartheta k \alpha_k^2)(1 - k \alpha_j^2)] \operatorname{sh} \alpha_i \operatorname{ch} \alpha_j \operatorname{ch} \alpha_k + \\
& + \alpha_1^2 \alpha_3^2 \alpha_5^2 \operatorname{sh} \alpha_1 \operatorname{sh} \alpha_3 \operatorname{sh} \alpha_5 \sum_{i,j,k} \alpha_i^2 \alpha_j^2 (\alpha_i^2 - \alpha_j^2)^2 (1 - \vartheta k \alpha_k^2)(1 - k \alpha_k^2) = 0 \quad (2.18)
\end{aligned}$$

(the indices i, j, k (1, 3, 5) are cyclically permuted under the summation signs three times).

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