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Abstract

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MATHEMATICAL PHYSICS

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ON THE COMPLETENESS OF THE AMPLITUDES OF PERTURBATION THEORY IN THE SPACE OF AMPLITUDES

(Presented by Academician N. N. Bogolyubov on 4 V 1966)

1. Lehmann ⁽¹⁾ established that the scattering amplitudes for processes $1+2 \rightarrow 3+4$ are holomorphic functions of the cosine of the scattering angle inside a certain ellipse. It is therefore expedient to introduce the following definition.

A linear space consisting of functions of the complex variable z , holomorphic inside an ellipse E with foci at the points $(-1, 0)$, $(1, 0)$ and with major semiaxis $\eta > 1$, will be called the **space of amplitudes** and denoted by H_0 (the number η is determined according to Lehmann's theorem).

For some models, for example nonrelativistic scattering by the Yukawa potential, $V(r) = e^{-\mu r}/r$, $\mu = 1$, and relativistic scattering described by the Bethe-Salpeter equation in the ladder approximation, the scattering amplitudes, for a sufficiently small value of the interaction constant g , can be represented in the form

$$f(s, \cos \theta, g) = \sum_{n=0}^{\infty} g^{n+1} f^n(s, \cos \theta), \quad (1)$$

where the functions $f^n(s, \cos \theta)$ are holomorphic in $\cos \theta$ inside the corresponding Lehmann ellipses, and the series (1), for fixed s and $\cos \theta$ from a certain domain, converges uniformly in g in some neighborhood of zero. The functions $f^n(s, \cos \theta) = f^n(s, z)$ will be called **amplitudes of perturbation theory**. In the relativistic case (the Bethe-Salpeter equation), the functions $f^n(s, z)$ are contributions from ladder Feynman diagrams (with equal internal and external masses $\mu = 1$).

In the present note the functions $f^n(s, z)$ are considered for fixed $s > 0$ in the case of nonrelativistic scattering and for $3 < s < 4$ in the case of relativistic scattering. Under these restrictions on s , the functions $f^n(s, z)$ are holomorphic

inside ellipses E_n with major semiaxis equal to $\text{ch}(n+1)\alpha$, $\text{ch } \alpha = 1 + 1/2s = \eta$, in the nonrelativistic case, and with major semiaxis equal to $|1 + 2(n+1)^2/(s-4)|$, $\text{ch } \alpha = 1 + 2/(s-4)$, in the relativistic case.

2. Introduce the functions

$$\tilde{f}^n(s, z) = \frac{1}{M_n L_n} f^n(s, z),$$

where $M_n = \max |f^n(s, z)|$ in the ellipse with major semiaxis equal to $\text{ch}(n+1)(\alpha - \varepsilon)$ in the nonrelativistic case and to $|1 + 2(n+1)^2/(s-4)| - \varepsilon$ in the relativistic case; $\varepsilon > 0$ is a sufficiently small number, and L_n is the length of the ellipse.

Consider, in addition, the functions

$$f_l(s, g) = \frac{1}{2} \int_{-1}^1 P_l(z) f(s, z, g) dz,$$

where $P_l(z)$ is a Legendre polynomial. These functions are meromorphic in g in the complex plane and, as l increases, decrease exponentially, $f_l(s, g) \sim e^{-\alpha l}$ ^(2,3).

The following two lemmas are valid; they characterize the strengthened linear independence of the functions $\tilde{f}^n(s, z)$ and $f_l(s, g)$.

Lemma 1. The expression $\sum_{n=0}^{\infty} a_n \tilde{f}^n(s, z)$, where $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, can be equal to zero for $-1 \leq z \leq 1$ only in the case when all $a_n = 0$.

The proof of this lemma follows from the fact that the functions $f^n(s, z)$ are holomorphic in ever-increasing domains, and their singularities cannot compensate one another.

Lemma 2. The expression

$$\sum_{l=0}^{\infty} a_l \sqrt{2(2l+1)} f_l(s, g), \quad \text{where } \sum_{l=0}^{\infty} |a_l|^2 < \infty$$

can be equal to zero for all g only in the case when all $a_l = 0$ (for almost all s from the interval $s > 0$ in the relativistic case and $3 < s < 4$ in the nonrelativistic case—only for such s are all subsequent results valid).

The proof of Lemma 2 is carried out according to the same scheme as the proof of Lemma 1; here one interesting property of the functions $f_l(s, y)$ is used, namely that for $l_1 \neq l_2$ all the poles of the functions $f_{l_1}(s, g)$ and $f_{l_2}(s, g)$ cannot coincide.

3. Let H denote the Hilbert space of functions square-summable on the interval $(-1, 1)$. With the aid of Lemmas 1 and 2 one can prove Theorems 1 and 2.

Theorem 1. The system of functions $f^n(s, z)$, $n = 0, 1, 2, \dots$, is complete in H .

Let us also define a sequence of functions $\varphi_l(s, z)$ by the formulas

$$\varphi_l(s, z) = \sum_{n=0}^{\infty} \tilde{f}_l^n(s) \sqrt{2(2l+1)} \sqrt{\frac{2n+1}{2}} P_n(z),$$

$$\tilde{f}_l^n(s) = \frac{1}{2} \int_{-1}^1 \tilde{f}^n(s, z) P_l(z) dz.$$

Theorem 2. The system of functions $\varphi_l(s, z)$, $l = 0, 1, 2, \dots$, is complete in H .

Since $H_0 \subset H$, the systems of functions $f^n(s, z)$ and $\varphi_l(s, z)$ are complete in the sense of the metric of H in the space of amplitudes H_0 .

Denote by $E_{1/2}$ the ellipse with foci at the points $(-1, 0)$, $(1, 0)$ and with major semiaxis equal to $\text{ch}((\alpha - \varepsilon)/2)$. Using Hilbert spaces of functions holomorphic inside the ellipse $E_{1/2}$ ⁽⁴⁾, one can approach the question of the completeness of the system of functions $f^n(s, z)$ in H_0 in the sense of uniform convergence.

Theorem 3. The system of functions $f^n(s, z)$ is complete in H_0 in the sense of uniform convergence inside the ellipse $E_{1/2}$.

For the proof of these theorems, the methods developed by A. N. Markushevich ⁽⁵⁾ and M. G. Khaplanov ⁽⁶⁾ are used.

Thus, the proof of Theorems 1 and 2 is carried out according to the following scheme. The operator $A = (a_{ln})$, $a_{ln} = \tilde{f}_l^n \sqrt{2(2l+1)}$, is constructed; it is defined in H and is completely continuous. According to Lemmas 1 and 2, the operators A and A^* have no eigenvalues equal to zero. The functions $\tilde{f}^n(s, z)$ and $\varphi_l(s, z)$ are defined with the aid of the operators A and A^* as follows:

$$\tilde{f}^n = A e_n, \quad \varphi_l = A^* e_l, \quad e_l = \sqrt{\frac{2l+1}{2}} P_l,$$

and since the system of functions e_l is complete in H , and the operators A and A^* have no eigenvalues equal to zero, the systems of functions

$$\tilde{f}^n(s, z) = \frac{1}{M_n L_n} f^n(s, z)$$

and $\varphi_l(s, z)$ are complete in H .

Theorem 3 is proved according to the same scheme.

The following question remains open: whether the system of functions $f^n(s, z)$ is a basis in H and in H_0 .

4. It seems to us that the results presented in this note on the completeness of perturbation-theory amplitudes in the space of amplitudes may prove useful in determining the scattering amplitude in specific problems. Since

the scattering amplitude can be represented through linear combinations of the functions $f^n(s, z)$, in order to determine the scattering amplitude it is sufficient to compute the coefficients of $f^n(s, z)$ in the expansion of the scattering amplitude in terms of $f^n(s, z)$.

It is natural to suppose that amplitudes for various processes (with the participation of an arbitrary number of particles) can be expanded in terms of contributions from a definite sequence of Feynman diagrams of those same processes.

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Note: Figure translations are in progress. See original paper for figures.

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