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1967

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Abstract

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UDC 517.512.2

MATHEMATICS

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REPRESENTATIONS OF THE QUATERNION GROUP OVER A NONCONNECTED LOCALLY COMPACT NONDISCRETE FIELD

1. Let E be the division ring of quaternions over a nonconnected locally compact nondiscrete field $\mathbf{K}$ (the definitions are given below); G the group of all quaternions with modulus equal to one; PG the quotient group of the group G by its center. It is assumed of the field $\mathbf{K}$ that its characteristic is not equal to 2 and that it does not contain the subfield Q_2 of 2-adic numbers. The paper gives a complete description of the irreducible representations of the group PG and computes the traces of these representations. We note that an analogous description of irreducible representations holds for the group SG of quaternions of norm 1 and for the group $PE^* = E^*/Z$, where E^* is the full group of quaternions, Z its center*.

We give the basic definitions and notation. The division ring of quaternions E over $\mathbf{K}$ is the set of elements of the form

$$x = a_0 + a_1i + a_2j + a_3k, \quad (1)$$

where $a_\nu \in \mathbf{K}$. Addition of elements (1) is carried out componentwise, and multiplication in accordance with the following table:

$$\begin{aligned} i^2 = \varepsilon, \quad j^2 = \mathfrak{p}, \quad k^2 = -\varepsilon\mathfrak{p}, \quad ij = -ji = k, \quad jk = -kj = -\mathfrak{p}i, \\ ki = -ik = -\varepsilon j. \end{aligned} \quad (2)$$

Here ε is a fixed element of $\mathbf{K}$ with norm 1, not a square in $\mathbf{K}$; $\mathfrak{p}$ is a fixed generating element of the prime ideal P in the ring O of integers of $\mathbf{K}^{**}$. The quaternion $\bar{x} = a_0 - a_1i - a_2j - a_3k$ is called conjugate to the quaternion (1); the product $N(x) = x\bar{x}$ is called the norm of the quaternion x . Since $N(x) = a_0^2 - \varepsilon a_1^2 - \mathfrak{p}a_2^2 + \varepsilon\mathfrak{p}a_3^2$, we have $N(x) \in \mathbf{K}$. The real number $|x| = \|N(x)\|_K^{1/2}$, where $\|a\|_K$ is the norm of an element $a \in \mathbf{K}$, is called the modulus of the quaternion x . Introduce the notation: $q = \|\mathfrak{p}\|_K^{-1}$. Then we have: $|x| = \|x\|_K$,

if $x \in \mathbf{K}$; $|i| = 1$, $|j| = |k| = q^{-1/2}$. Hence it follows that $|x|$ can take only the values 0 and $q^{n/2}$, $n = 0, \pm 1, \pm 2, \dots$

2. Rank of an irreducible representation. The groups G and $PG = G/Z$ are compact, and therefore all their irreducible representations are finite-dimensional. Instead of constructing irreducible representations of PG , we shall construct irreducible representations $T(x)$ of the group G that are equal

* These three groups have a closely related structure. Namely, the following exact sequences of homomorphisms hold: $O \rightarrow PG \rightarrow PE^* \rightarrow Z/2Z \rightarrow O$, $O \rightarrow Z/2Z \rightarrow SG \rightarrow PSG \rightarrow O$, $O \rightarrow PSG \rightarrow PG \rightarrow Z/2Z \rightarrow O$, where $Z/2Z$ is the cyclic group of second order, and PSG is the quotient group of the group SG by its center.

** We recall the definition of the quaternion algebra E over an arbitrary commutative ring A with identity (1). The algebra E is the set of elements of the form (1), where $a_\nu \in A$; multiplication is defined by the following table: $i^2 = a$, $j^2 = b$, $k^2 = -ab$, $ij = -ji = k$, $jk = -kj = -bi$, $ki = -ik = -aj$. Here a, b are fixed elements of A for which $ab \neq 0$; the algebra E depends, generally speaking, on the choice of these elements. In the case $A = \mathbf{K}$ there are only two nonisomorphic quaternion algebras—the algebra of all matrices of order two over $\mathbf{K}$ and the division algebra described above. We note that the representations of the group $SL(2, \mathbf{K})$ of matrices of order 2 over $\mathbf{K}$ were completely described in (2).

to the identity operator on its center Z ; obviously, these two problems are equivalent.

Let us specify in G a decreasing chain of normal divisors of finite index: $G = G_0 \supset G_{1/2} \supset G_1 \supset G_{3/2} \supset \dots$; G_ν is the subgroup of all quaternions $x \in G$ of the form $x = a + x'$, where $a \in \mathbf{K}$, $|x'| \leq q^{-\nu}$. We have: $\bigcap_\nu G_\nu = Z$; therefore, if an irreducible representation $T(x)$ of the group G is equal to 1 on Z , then $T(x) = 1$ on G_ν for sufficiently large ν . We shall call the **rank** of the representation $T(x)$ the smallest ν for which $T(x) = 1$ on G_ν . Obviously, the description of irreducible representations of rank ν is equivalent to the description of irreducible representations of the finite quotient group G/G_ν that are not identically equal to the identity on $G_{\nu-1/2}/G_\nu$. The only representation of rank 0 is the identity representation. Further, since $G/G_{1/2}$ is a commutative group of order $q + 1$, all representations of rank 1 are one-dimensional, and their number is q . Each of these representations is given by the mapping

$$a_0 + a_1 i + a_2 j + a_3 k \rightarrow \pi(a_0 + \sqrt{\varepsilon} a_1),$$

where π is a multiplicative character on $\mathbf{K}(\sqrt{\varepsilon})$ such that $\pi \neq 1$, $\pi(\alpha) \equiv 1$ for $\alpha \in \mathbf{K}$, and $\pi(1 + \alpha) \equiv 1$ for $|\alpha| < 1$. Below we give a description of the irreducible representations of ranks $\nu > 1/2$.

3. Description of irreducible representations of rank ν , where ν is an integer.

a) **Representations of the first series.** Let us specify the commutative subalgebra U_ρ of quaternions of the form $\alpha = a + bj$, $a, b \in \mathbf{K}$. Obviously, every quaternion x can be represented uniquely in the form $x = \alpha + \beta i$, where $\alpha, \beta \in U_\rho$. Introduce the subgroup $A'_\nu \subset G$ of quaternions $x = \alpha + \beta i \in G$ for which $|\beta| \leq q^{-\nu/2}$. Let us specify one-dimensional representations of the group A'_ν . Let $\pi(\alpha)$ be an arbitrary character on the multiplicative group U_ρ^* of elements $\alpha \in U_\rho$ with modulus $|\alpha| = 1$, satisfying the following conditions:

- 1) $\pi(\alpha) \equiv 1$ for $\alpha \in \mathbf{K}$; 2) the rank of π is ν ; this means that $\pi(1 + \alpha) \equiv 1$ for $|\alpha| \leq q^{-\nu}$, $\pi(1 + \alpha) \neq 1$ for $|\alpha| = q^{-\nu+1/2}$.

Then the mapping

$$\alpha + \beta i \rightarrow \pi(\alpha) \quad (3)$$

is a homomorphism of the subgroup A'_ν , i.e., it gives its one-dimensional representation. Let $T'_\pi(x)$ be the representation of the group G induced by the representation (3) of the subgroup A'_ν . From the definition it follows readily that the rank $T'_\pi = \nu$. The representations T'_π will be called the first series of representations of rank ν .

b) **Representations of the second series.** Instead of U_ρ , specify the subalgebra $U_{\varepsilon\rho}$ of quaternions $\alpha = a + bj'$, where $j' = a_1j + a_2k$ is a quaternion satisfying the relation $j'^2 = \varepsilon\rho$. Obviously, every quaternion x can be represented uniquely in the form $x = \alpha + \beta i$, where $\alpha, \beta \in U_{\varepsilon\rho}$. Introduce the subgroup $A''_\nu \subset G$ of quaternions $x = \alpha + \beta i \in G$, $\alpha, \beta \in U_{\varepsilon\rho}$, for which $|\beta| \leq q^{-\nu/2}$. Let us specify one-dimensional representations of the group A''_ν by formula (3), where $\pi(\alpha)$ is an arbitrary character on $U_{\varepsilon\rho}^*$ satisfying conditions 1) and 2). Denote by $T''_\pi(x)$ the representations of the group G induced by these one-dimensional representations of the subgroup A''_ν . From the definition it follows that the rank $T''_\pi = \nu$. The representations T''_π will be called the second series of representations of rank ν .

Theorem 1. 1) The representations T'_π, T''_π are irreducible; 2) two representations T_{π_1}, T_{π_2} are equivalent if and only if they belong to one and the same series and either $\pi_1 = \pi_2$, or $\pi_1 = \pi_2^{-1}$; 3) $\dim T'_\pi = \dim T''_\pi = (q+1)q^{\nu-1}$; 4) the number of pairwise inequivalent representations in each of the series is equal to $\frac{q-1}{2}q^{\nu-1}$.

4. Description of irreducible representations of rank $\nu = 2m + 1/2$, $m = 0, 1, 2, \dots$ Let us specify the commutative subalgebra U_ε of quaternions of

the form $\alpha = a + bi$, $a, b \in \mathbf{K}$. Obviously, every quaternion x

uniquely representable in the form $x = \alpha + \beta j$, where $\alpha, \beta \in U_\varepsilon$. Introduce the subgroup $A_\nu \subset G$ of quaternions $x = \alpha + \beta j \in G$, $\alpha, \beta \in U_\varepsilon$, for which $|\beta| \leq q^{-m}$. Define one-dimensional representations of the group A_ν . Let $\pi(\alpha)$ be an arbitrary character on the multiplicative group U_ε^* of elements $\alpha \in U_\varepsilon$ with modulus $|\alpha| = 1$, satisfying conditions 1) and 2) of § 3. Then the mapping

$$x = \alpha + \beta j \rightarrow \pi(\alpha) \quad (4)$$

defines a one-dimensional representation of the group A_ν .

Denote by $T_\pi(x)$ the representations of the group G induced by these representations of the subgroup A_ν . It follows from the definition that the rank $T_\pi = \nu$.

Theorem 2. 1) The representations T_π are irreducible and pairwise inequivalent; 2) $\dim T_\pi = q^{\nu-1/2}$; 3) the number of representations T_π is equal to $(q^2 - 1)q^{\nu-3/2}$.

5. Description of irreducible representations of rank $\nu = 2m + 3/2$, $m = 0, 1, 2, \dots$

Introduce the subgroup $A_\nu \subset G_{1/2}$ of quaternions $x = \alpha + \beta j$, $\alpha, \beta \in U_\varepsilon$, for which $|\beta| \leq q^{-m}$, $|\alpha - \bar{\alpha}| < 1$, $|\beta - \bar{\beta}| < q^{-m}$. Let $\pi(\alpha)$ be an arbitrary character on $U_\varepsilon^* \cap A_\nu$ satisfying conditions 1) and 2) of § 3. Then the mapping (4) is a one-dimensional representation of the subgroup A_ν . Denote by $T_\pi(x)$ the representation of the subgroup $G_{1/2}$ induced by this one-dimensional representation. By definition, the representation T_π acts in the space H_π of functions $f(y)$ on $G_{1/2}$ satisfying the following condition: $f(xy) = \pi(x)f(y)$ for any $x \in A_\nu$ ($\pi(x)$ is the function on A_ν given by the mapping (4)). The operator $T_\pi(y)$ is given by the following formula: $T_\pi(y_0)f(y) = f(y y_0)$.

We now extend each of the representations T_π to a representation of the entire group G . Note that there exists an element $x_0 \in U_\varepsilon^*$ of order $q^2 - 1$ such that $G = \{x_0\} \cdot G_{1/2}$ and $\{x_0\} \cap G_{1/2} \subset Z$, where Z is the center of the group G ($Z \subset G_{1/2}$). From the definition of x_0 it follows that $x_0^{q+1} \in Z$ and that any element $x \in G$ is uniquely representable in the form $x = x_0^s y$, where $s = 0, 1, \dots, q$; $y \in G_{1/2}$. Associate to the element x_0 an operator B_{x_0} on H_π :

$$(B_{x_0} f)(y) = \sum_{x \in A_\nu / G_\nu} \pi^{-1}(x) f(x_0^{-1} x y x_0).$$

The following assertions are valid: 1) B_{x_0} is a nondegenerate operator on H_π ; 2) $B_{x_0} T_\pi(y) = T_\pi(x_0 y x_0^{-1}) B_{x_0}$ for any $y \in G_{1/2}$; 3) $B_{x_0}^{q+1} = \lambda E$, where E is the identity operator. Let $\mu_1, \dots, \mu_{q+1}$ be the set of roots of the equation $\lambda t^{q+1} = 1$.

To each $k = 1, \dots, q + 1$ assign the following representation $T_{\pi,k}$ of the group G : $T_{\pi,k}$ acts in the space H_π of the representation T_π of the subgroup $G_{1/2}$ defined above; the operator $T_{\pi,k}(x)$, corresponding to the element $x = x_0^s y$, where $s = 0, 1, \dots, q$; $y \in G_{1/2}$, is determined by the formula

$$T_{\pi,k}(x) = (\mu_k B_{x_0})^s T_\pi(y).$$

(The verification that $T_{\pi,k}$ is a representation is elementary.) From the construction it follows easily that the rank $T_{\pi,k} = \nu$.

Theorem 3. 1) The representations $T_{\pi,k}$ are irreducible and pairwise inequivalent; 2) $\dim T_{\pi,k} = q^{\nu-1/2}$; 3) the number of representations $T_{\pi,k}$ is equal to $(q^2 - 1)q^{\nu-3/2}$.

6. Completeness theorem

The representations described above exhaust all irreducible representations of the group G that are equal to the identity on its center.

Proof. Consider the representations of fixed rank ν described above. In Theorems 1, 2, 3 the number of these representations and their dimensions are indicated. Hence, by direct calculation, one can verify that the sum of the squares of the dimensions of these representations is equal to the difference $\dim G/G_\nu - \dim G/G_{\nu-1/2}$. Consequently, by the Bernshtein theorem—

for finite groups, these representations exhaust all irreducible representations of rank ν .

7. Formulas for the characters of irreducible representations. It is easy to verify that in each class of conjugate elements of the group G there is an element from the union $U = U_\varepsilon^* \cup U_{\mathfrak{p}}^* \cup U_{\varepsilon\mathfrak{p}}^*$. Below we give expressions for $\text{tr } T(x)$ on the set U .

a) ν is an integer, T_π is an irreducible representation of rank ν , belonging to the first series. If $\alpha \in U_{\mathfrak{p}}^*$, $|\alpha - \bar{\alpha}| > q^{-\nu+1/2}$, then

$$\text{tr } T_\pi(\alpha) = \text{sign}_{\mathfrak{p}}(-1)^\nu \cdot c_\pi |\alpha - \bar{\alpha}|^{-1} (\text{sign}_{\mathfrak{p}}(j^{-1}(\alpha^2 - \bar{\alpha}^2))\pi(\alpha) + \text{sign}_{\mathfrak{p}}(j^{-1}(\bar{\alpha}^2 - \alpha^2))\pi(\bar{\alpha})).$$

If $|a - \bar{a}| = q^{-\nu+1/2}$, $\alpha = a + \mathfrak{p}^{\nu-1}bj \in U_{\mathfrak{p}}^*$, then

$$\text{tr } T_\pi(\alpha) = \frac{\dim T_\pi}{q^2 - 1} \sum \pi \left(a + \mathfrak{p}^{\nu-1}b \frac{x^2 + \varepsilon y^2}{x^2 - \varepsilon y^2} j \right),$$

where the summation is over $x, y \in O/P$, $(x, y) \neq 0$.* If $|\alpha - \bar{\alpha}| > q^{-\nu}$, $\alpha \in U_\varepsilon^* \cup U_{\varepsilon\mathfrak{p}}^*$, then $\text{tr } T_\pi(\alpha) = 0$. If $|\alpha - \bar{\alpha}| \leq q^{-\nu}$, then

$$\text{tr } T_\pi(\alpha) = \dim T_\pi = (q + 1)q^{\nu-1}.$$

Here the notation is as follows: $\text{sign}_\tau x = 1$ when $x \in K$ is representable in the form $x = a^2 - \tau b^2$, $a, b \in K$; $\text{sign}_\tau x = -1$ otherwise; $c_\pi = q^{-1/2} \sum \pi(1 - 2\varepsilon \mathfrak{p}^{\nu-1} x^2 j)$, the summation being over $x \in O/P$; it is easy to verify that $c_\pi = \pm 1$, if $-1 = a^2$, $a \in K$; and $c_\pi = \pm \sqrt{-1}$ otherwise.

- b) The formulas for the characters of representations of the second series are obtained by replacing $\mathfrak{p}$ by $\varepsilon \mathfrak{p}$ and j by j' .
- c) $\nu = 2m + 1/2$. If $|\alpha - \bar{\alpha}| > q^{-\nu}$, $\alpha \in U_\varepsilon^*$, then

$$\text{tr } T_\pi(\alpha) = |\alpha - \bar{\alpha}|^{-1} \text{sign}_\varepsilon(i^{-1}(\alpha - \bar{\alpha}))\pi(\alpha).$$

If $|\alpha - \bar{\alpha}| > q^{-\nu}$, $\alpha \in U_\mathfrak{p}^* \cup U_{\varepsilon \mathfrak{p}}^*$, then $\text{tr } T_\pi(\alpha) = 0$. If $|\alpha - \bar{\alpha}| \leq q^{-\nu}$, then

$$\text{tr } T_\pi(\alpha) = \dim T_\pi = q^{\nu-1/2}.$$

- d) $\nu = 2m + 3/2$. If $|\alpha - \bar{\alpha}| > q^{-\nu}$, $\alpha \in U_\varepsilon^*$, then

$$\text{tr } T_{\pi,k}(\alpha) = -|\alpha - \bar{\alpha}|^{-1} \text{sign}_\varepsilon(i^{-1}(\alpha - \bar{\alpha}))\pi_k(\alpha);$$

$\pi_k(\alpha)$ is a certain character on U_ε^* such that $\pi_k(\alpha) = \pi(\alpha)$ when $\alpha \in U_\varepsilon^* \cap A_{\nu}^{**}$. If $|\alpha - \bar{\alpha}| > q^{-\nu}$, $\alpha \in U_\mathfrak{p}^* \cup U_{\varepsilon \mathfrak{p}}^*$, then $\text{tr } T_{\pi,k}(\alpha) = 0$. If $|\alpha - \bar{\alpha}| \leq q^{-\nu}$, then

$$\text{tr } T_{\pi,k}(\alpha) = \dim T_{\pi,k} = q^{\nu-1/2}.$$

Received
27 VII 1967

CITED LITERATURE

1. Bourbaki, *Algebra*, Moscow, 1962.
2. I. M. Gel'fand, M. I. Graev, I. I. Pyatetskii-Shapiro, *Representation Theory and Automorphic Functions*, "Nauka," 1966.

* Let us note that $\chi_\pi(x) = \pi(1 + \mathfrak{p}^{\nu-1} x j)$ is an additive character on $F_q = O/P$; the formula for the trace can be given the form

$$\text{tr } T_\pi(\alpha) = \frac{\dim T_\pi}{q^2 - 1} \sum \chi_\pi \left(\frac{b}{a} \frac{x^2 + \varepsilon y^2}{x^2 - \varepsilon y^2} \right) = \frac{\dim T_\pi}{q + 1} \sum_{t \bar{t} = 1} \chi_\pi \left(\frac{b}{2a} (t + \bar{t}) \right),$$

where $t = x + \sqrt{\varepsilon} y$.

** Let us note that any two characters $\pi_k(\alpha)$, $\pi_l(\alpha)$ ($k, l = 1, \dots, q + 1$) are related by

$$\mu_k^{-1} \pi_k(x_0) = \mu_l^{-1} \pi_l(x_0)$$

(the notation is that of §5). Hence it follows that they are uniquely expressible in terms of one another.

Note: Figure translations are in progress. See original paper for figures.

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