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Abstract

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MATHEMATICS

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ESTIMATE OF THE DIFFERENCE OF TWO MATRICANTS

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1. Notation. The word matrix means a square matrix of fixed order n , whose elements are complex numbers. If A is a matrix, then $[A]_{\mu\nu}$ ($\mu, \nu = 1, \dots, n$) are its elements; $|A|$ denotes the matrix whose elements are the moduli of the corresponding elements of A ; A^+ denotes the matrix whose diagonal (off-diagonal) elements are the real parts (moduli) of the corresponding elements of A ; $|A|$ is called the modulus of the matrix A . The symbol E denotes the identity matrix; the symbol I denotes the matrix all of whose elements are equal to one. A matrix-function is a matrix whose elements are functions of the real parameter t . If $A = A(t)$ is a matrix-function, then

$$A^+(t) \stackrel{\text{def}}{=} (A(t))^+.$$

Inequalities between matrices, convergence of a sequence of matrices and of a matrix series, and also continuity and differentiability of matrix-functions are understood elementwise. The letters A, B, C, D, P, Q (also with indices) denote matrices or, if this is stipulated, matrix-functions; the latter are always assumed to be continuous. It is useful to note that

$$|A + B| \leq |A| + |B|, \quad |AB| \leq |A| |B|.$$

A norm is introduced in the set of matrices, satisfying the three axioms of the norm of a linear complex space and the conditions

$$\|AB\| \leq \|A\| \|B\|, \quad \|E\| = 1.$$

It is known that

$$\lim_{k \rightarrow \infty} \|A_k - A\| = 0$$

is equivalent to

$$\lim_{k \rightarrow \infty} A_k = A$$

(see ⁽¹⁾, Chap. 1, § 13, no. 3). The norm is used below only in the formulation and proof of Theorem 2 (paras. 4, 8, 10; see also paras. 12, 13).

2. Consider the linear homogeneous system of differential equations (in vector notation)

$$dx/dt = A(t)x, \quad (1)$$

where $A(t)$ is a matrix-function on the interval $[t_0, T]$. The symbol

$${}_{\xi}^t \Omega A$$

(where $t_0 \leq \xi$, $t \leq T$) denotes the matricant of the matrix-function $A(t)$, i.e. the matrix-function whose columns are the solutions of system (1) satisfying the condition

$${}_{\xi}^{\xi} \Omega A = E$$

(for the properties of the matricant, see ⁽²⁾, Chap. XIV, §§ 5, 6). If, along with system (1), the system $dx/dt = B(t)x$ is also considered, then it is often necessary to estimate

$${}_{t_0}^t \Omega B - {}_{t_0}^t \Omega A.$$

Formulation of the results.

3. Theorem 1 (estimate of the modulus of the difference of matricants). *Let A, B, C, D be matrix-functions on $[t_0, T]$, and let*

$$A^+(t) \leq C(t), \quad |B(t) - A(t)| \leq D(t) \quad \text{for } t_0 \leq t \leq T. \quad (2)$$

Then:

a)

$$\left| {}_{t_0}^t \Omega B - {}_{t_0}^t \Omega A \right| \leq {}_{t_0}^t \Omega (C + D) - {}_{t_0}^t \Omega C \quad \text{for } t_0 \leq t \leq T.$$

b) If, moreover, the matrices $C(t_1), C(t_2), D(t_1), D(t_2)$ commute pairwise for $t_0 \leq t_1, t_2 \leq T$, then

$$\left| \Omega_{t_0}^t B - \Omega_{t_0}^t A \right| \leq \exp \int_{t_0}^t C(u) du \cdot \left\{ \exp \int_{t_0}^t D(u) du - E \right\} \quad \text{for } t_0 \leq t \leq T.$$

c) In particular, if

$$C(t) = q(t)I + \alpha(t)E, \quad D(t) = \delta(t)I \quad \text{for } t_0 \leq t \leq T,$$

where the scalar functions $q(t), \alpha(t), \delta(t)$ are continuous on $[t_0, T]$, then

$$\left| \Omega_{t_0}^t B - \Omega_{t_0}^t A \right| \leq \exp \int_{t_0}^t \alpha du \cdot \exp \left(\int_{t_0}^t q du I \right) \cdot \left\{ \exp \left(\int_{t_0}^t \delta du I \right) - E \right\} \quad \text{for } t_0 \leq t \leq T.$$

4. **Theorem 2 (estimate of the norm of the difference of multiplicative integrals).** Suppose there exist real functions $g(t), M(t), L(t), \delta(t)$, continuous on $[t_0, T]$ and satisfying the conditions

$$\|\Omega_\xi^t A\| \leq M(t)L(\xi) \exp \int_\xi^t g(u) du \quad \text{for } t_0 \leq \xi \leq t \leq T, \quad (3)$$

$$\|B(t) - A(t)\| \leq \delta(t) \quad \text{for } t_0 \leq t \leq T. \quad (4)$$

Then

$$\|\Omega_{t_0}^t B - \Omega_{t_0}^t A\| \leq M(t)L(t_0) \exp \int_{t_0}^t g du \cdot \left\{ \exp \int_{t_0}^t ML\delta du - 1 \right\} \quad \text{for } t_0 \leq t \leq T. \quad (5)$$

Proofs.

5. Suppose $|A_k| \leq P_k$, $|B_k - A_k| \leq D_k$ for $k = 1, \dots, N$. Then

$$\begin{aligned} & |B_N B_{N-1} \cdots B_1 - A_N A_{N-1} \cdots A_1| \leq \\ & \leq (P_N + D_N)(P_{N-1} + D_{N-1}) \cdots (P_1 + D_1) - P_N P_{N-1} \cdots P_1. \end{aligned}$$

Proof. For $N = 1$ this is obvious. Noting that

$$\begin{aligned} & B_{N+1} B_N \cdots B_1 - A_{N+1} A_N \cdots A_1 = \\ & = (B_{N+1} - A_{N+1}) B_N \cdots B_1 + A_{N+1} (B_N \cdots B_1 - A_N \cdots A_1), \end{aligned}$$

we apply induction on N .

6. Suppose R is real, S is a real diagonal matrix, Δ is a nonnegative number, and

$$[R]_{\mu\mu} \geq -\frac{1}{2}, \quad |[S]_{\mu\mu}| \leq \Delta \quad \text{for } \mu = 1, \dots, n;$$

$$[R]_{\mu\nu} \geq 0 \quad \text{for } \mu, \nu = 1, \dots, n; \mu \neq \nu.$$

Then

$$|E + R + iS| \leq (1 + 2\Delta^2)(E + R).$$

Proof. By elementary calculations.

7. Proof of Theorem 1. Define the diagonal matrix-function S by the formulas

$$[S(t)]_{\mu\mu} \stackrel{\text{def}}{=} \text{Im}[A(t)]_{\mu\mu}, \quad \text{for } \mu = 1, \dots, n.$$

Find a nonnegative number Δ and a natural number N_0 such that

$$|[S(t)]_{\mu\mu}| \leq \Delta, \quad \frac{T - t_0}{N_0} [A^+(t)]_{\mu\mu} \geq -\frac{1}{2} \quad \text{for } \mu = 1, \dots, n; \quad t_0 \leq t \leq T. \quad (6)$$

Fix t , $t_0 \leq t \leq T$, and a natural number $N > N_0$. Put

$$h \stackrel{\text{def}}{=} \frac{t - t_0}{N}, \quad t_k \stackrel{\text{def}}{=} t_0 + kh \quad \text{for } k = 1, \dots, N. \quad (7)$$

Finally, set $A_k \stackrel{\text{def}}{=} A(t_k)$ for $k = 1, \dots, N$, and analogously for matrix-functions B, C, D, S . By item 6, (6), and (7),

$$\begin{aligned} |E + hA_k| &= |E + hA_k^+ + ihS_k| \leq \\ &\leq (1 + 2h^2\Delta^2)(E + hA_k^+) \leq (1 + 2h^2\Delta^2)(E + hC_k) \\ &\quad \text{for } k = 1, \dots, N. \end{aligned} \quad (8)$$

By (8) and the second of inequalities (2),

$$\begin{aligned} |E + hB_k| &\leq |E + hA_k| + h|B_k - A_k| \leq |E + hA_k| + hD_k \leq \\ &\leq (1 + 2h^2\Delta^2)(E + h(C_k + D_k)) \quad \text{for } k = 1, \dots, N. \end{aligned}$$

Replacing, in item 5, A_k by $E + hA_k$, B_k by $E + hB_k$, P_k by $(1 + 2h^2\Delta^2) \times (E + hC_k)$, and D_k by $(1 + 2h^2\Delta^2)hD_k$, we obtain

$$\begin{aligned} &|(E + hB_N) \dots (E + hB_1) - (E + hA_N) \dots (E + hA_1)| \leq \\ &\leq (1 + 2h^2\Delta^2)^N \{(E + h(C_N + D_N)) \dots (E + h(C_1 + D_1)) - \\ &\quad - (E + hC_N) \dots (E + hC_1)\}. \end{aligned}$$

Letting $N \rightarrow \infty$, we obtain a); b) follows from a) ((2), p.386); c) follows from b).

8.

$$\|\Omega_{\xi}^t A\| \leq \exp \int_{\xi}^t \|A(u)\| du \quad \text{for } t_0 \leq \xi \leq t \leq T.$$

Proof. This follows from the representation of the matriciant as the limit of an integral product ((2), *Ch.XIV, formula(49')*).

9. Let A and B be matrix-functions on $[t_0, T]$. Define the matrix-function D by the formula

$$D(t) \stackrel{\text{def}}{=} B(t) - A(t) \quad \text{for } t_0 \leq t \leq T. \quad (9)$$

Then

$$\Omega_{t_0}^t B = \Omega_{t_0}^t A + \int_{t_0}^t (\Omega_{\xi}^t A) D(\xi) (\Omega_{t_0}^{\xi} B) d\xi \quad \text{for } t_0 \leq t \leq T. \quad (10)$$

Proof. Let $L(t)$ denote the left-hand side, and $\Pi(t)$ the right-hand side of formula (10). Then a simple calculation shows that the matrix-functions L and Π are solutions of the Cauchy problem

$$\frac{dX}{dt} = A(t)X + D(t)\Omega_{t_0}^t B, \quad X(t_0) = E \quad (11)$$

(X is the unknown matrix-function, $t_0 \leq t \leq T$). By uniqueness of the solution of problem (11), we have $L(t) = \Pi(t)$ for $t_0 \leq t \leq T$, as was required to prove.

10. **Proof of Theorem 2.** Keeping (9), define the matrix-functions I_k, I_k^* ($k = 0, 1, 2, \dots$) by the formulas

$$I_0(t) \stackrel{\text{def}}{=} \Omega_{t_0}^t A, \quad I_0^*(t) \stackrel{\text{def}}{=} \Omega_{t_0}^t B, \quad (12)$$

$$I_{k+1} \stackrel{\text{def}}{=} \int_{t_0}^t (\Omega_{\xi}^t A) D(\xi) I_k(\xi) d\xi \quad \text{for } k = 0, 1, 2, \dots, \quad (13)$$

$$I_{k+1}^* \stackrel{\text{def}}{=} \int_{t_0}^t (\Omega_{\xi}^t A) D(\xi) I_k^*(\xi) d\xi \quad \text{for } k = 0, 1, 2, \dots, \quad (14)$$

where $t_0 \leq t \leq T$. Then

$$\Omega_{t_0}^t B = I_0(t) + \dots + I_k(t) + I_{k+1}^*(t) \quad \text{for } k = 0, 1, 2, \dots; \quad t_0 \leq t \leq T.$$

Indeed, for $k = 0$ this follows from (12), (14), and item 9; next we apply (using (10)) induction on k .

Denote by M some positive number such that

$$\|\Omega_{t_0}^t B\| \leq M \quad \text{for } t_0 \leq t \leq T. \quad (15)$$

Then

$$\|I_{k+1}^*(t)\| \leq M \exp \int_{t_0}^t \|A(u)\| du \cdot \frac{1}{(k+1)!} \left\{ \int_{t_0}^t \delta(u) du \right\}^{k+1} \quad (16)$$

for $k = 0, 1, 2, \dots; t_0 \leq t \leq T$.

Indeed, for $k = 0$ inequality (16) follows from item 8, (15), (14), and (12). Next we apply induction on k . From (16) it follows that $\|I_{k+1}^*(t)\| \rightarrow 0$ as $k \rightarrow \infty$, $t_0 \leq t \leq T$, and consequently,

$$\Omega_{t_0}^t B = \sum_{k=0}^{\infty} I_k(t) \quad \text{for } k = 0, 1, 2, \dots \quad (17)$$

Let us now note that

$$\|I_k(t)\| \leq M(t)L(t_0) \exp \int_{t_0}^t g(u) du \cdot \frac{1}{k!} \left\{ \int_{t_0}^t ML\delta du \right\}^k \quad (18)$$

for $k = 0, 1, 2, \dots; t_0 \leq t \leq T$.

Indeed, for $k = 0$ this is true by virtue of (12) and (3); we apply (using (13), (4), and (9)) induction on k . From (17) it follows that

$$\|\Omega_{t_0}^t B - \Omega_{t_0}^t A\| \leq \sum_{k=1}^{\infty} \|I_k(t)\| \quad \text{for } t_0 \leq t \leq T. \quad (19)$$

From (18) and (19) we obtain (5).

Additional Remarks

11. Let A and C be matrix-functions on $[t_0, T]$, with $A^+(t) \leq C(t)$ for $t_0 \leq t \leq T$. Then

$$|\Omega_{t_0}^t A| \leq \Omega_{t_0}^t C \quad \text{for } t_0 \leq t \leq T.$$

Proof. With the notation of item 7, (8) is true, which gives

$$|(E + hA_N) \cdots (E + hA_1)| \leq (1 + 2h^2 \Delta^2)^N (E + hC_N) \cdots (E + hC_1).$$

Let $N \rightarrow \infty$.

12. Theorem 1b) and item 11 generalize known results ((²), Ch. XIV, §6, VIII and §5, 5°; there—in different Condition (3) in Theorem 2 is fulfilled, for example, if $M(t) = L(t) = 1$, $g(t) = \gamma(A(t))$, where γ is the logarithmic norm of the matrix corresponding to the introduced matrix norm ((³), pp. 57–62).

13. The linear transformation $x = U(t)y$, where the matrix-function $U(t)$ has a continuous derivative and is nonsingular for $t_0 \leq t \leq T$, transforms system (1) into a system with matrix $Q(t)$, and

$$\Omega_\xi^t A = U(t) (\Omega_\xi^t Q) U^{-1}(\xi).$$

Hence it is easy to obtain a more general case of the fulfillment of (3) than in item 12.

14. If the matricant is defined as the sum of a series ((²), Ch. XIV, formula (10)), then in Theorems 1 and 2 it is sufficient, instead of continuity of the matrix-functions, to assume summability.

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Note: Figure translations are in progress. See original paper for figures.

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