

ON THE THEORY OF ζ -FUNCTIONS OF THREE VARIABLES

N. N. Sandakova

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Abstract

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MATHEMATICS

N. N. SANDAKOVA

ON THE THEORY OF ζ -FUNCTIONS OF THREE VARIABLES

(Presented by Academician S. L. Sobolev on 7 X 1966)

The formulation of the problem is described in the paper [1].

Let us denote

$$\zeta(A, m) = \sum_{x_i} \frac{1}{f^m(x_i)},$$

where $f(x_i)$ is a positive definite quadratic form in n variables, A is its matrix, $m > n/2$; the series is summed over all possible integral points.

In the present paper the following is proved.

Theorem. *The positive definite quadratic form*

$$f_0 = x^2 + y^2 + z^2 + xy + xz + yz,$$

corresponding to the densest packing of spheres in the space E^3 , gives a local minimum of the function $\zeta(A, m)$ for $m \geq 2$.

Proof. Varying the form f_0 , we obtain the form $f = f_0 + \bar{f}$, where

$$\bar{f} = \varepsilon_{11}x^2 + \varepsilon_{22}y^2 + \varepsilon_{33}z^2 + \varepsilon_{12}xy + \varepsilon_{13}xz + \varepsilon_{23}yz.$$

The form f_0 is transformed into itself by a group φ of integral unimodular transformations; the group φ , up to inversion, has order 24. As a fundamental domain Φ of the group φ we take the triangular angle determined by the inequalities $x - y > 0$, $y - z > 0$, $y + z > 0$. The initial point $B_1(x, y, z)$ will always be chosen in the closed domain Φ . The set K of integral points decomposes under the group φ into finitely many mutually disjoint sets of equivalent points. If the initial point $B_1(x, y, z) \in \Phi$, then, up to inversion, the set of points equivalent

to it will be

$$\begin{aligned}
& B_1(x, y, z), \quad B_2(y, z, x), \quad B_3(z, x, y), \quad B_4(x, z, y), \quad B_5(y, x, z), \quad B_6(z, y, x), \\
& B_7(x, y, -x - y - z), \quad B_8(y, -x - y - z, x), \quad B_9(-x - y - z, x, y), \\
& B_{10}(x, -x - y - z, y), \quad B_{11}(y, x, -x - y - z), \quad B_{12}(-x - y - z, y, x), \\
& B_{13}(-x - y - z, y, z), \quad B_{14}(y, z, -x - y - z), \quad B_{15}(z, -x - y - z, y), \\
& B_{16}(-x - y - z, z, y), \quad B_{17}(y, -x - y - z, z), \quad B_{18}(z, y, -x - y - z), \\
& B_{19}(x, -x - y - z, z), \quad B_{20}(-x - y - z, z, x), \quad B_{21}(z, x, -x - y - z), \\
& B_{22}(x, z, -x - y - z), \quad B_{23}(-x - y - z, x, z), \quad B_{24}(z, -x - y - z, x);
\end{aligned}$$

denote it by ω_1 . Depending on the choice on the boundary of Φ of the initial point $B_1(x, x, z)$, $B_1(x, y, y)$, $B_1(x, x, x)$, $B_1(x, x, -x)$, $B_1(x, x, 0)$, $B_1(x, 0, 0)$, we respectively obtain the sets $\omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7$, which are degenerate cases of the set ω_1 . The set ω_i ($i = 1, 2, \dots, 7$) consists of $l_i = 24/h$ points, where h is the order of the subgroup leaving the point $B_1(x, y, z)$ fixed. It is easy to compute

$$l_1 = 24, \quad l_2 = 12, \quad l_3 = 12, \quad l_4 = 4, \quad l_5 = 3, \quad l_6 = 12, \quad l_7 = 6.$$

Let B_{ik} be the k -th point of ω_i , $f_{ik} = f(B_{ik})$. We estimate the difference

$$\Delta\Sigma = \sum_{x,y,z} \frac{1}{f^m} - \sum_{x,y,z} \frac{1}{f_0^m}.$$

In view of the decomposition of the set K into the sets ω_i ($i = 1, 2, \dots, 7$), we have

$$\Delta\Sigma = \Delta_1\Sigma + \Delta_2\Sigma + \Delta_3\Sigma + \Delta_4\Sigma + \Delta_5\Sigma + \Delta_6\Sigma + \Delta_7\Sigma, \quad (1)$$

where

$$\Delta_i\Sigma = \sum_{x,y,z} \left(\frac{1}{f_{i1}^m} + \frac{1}{f_{i2}^m} + \dots + \frac{1}{f_{il_i}^m} - \frac{l_i}{f_0^m} \right), \quad B_1(x, y, z) \in \bar{\Phi}. \quad (1')$$

Formulas (1) and (1') make it possible to pass from summation over all possible integer points to summation only over those points which belong to the region $\bar{\Phi}$; therefore, in the further proof the problem is considered only in the region $\bar{\Phi}$.

In view of the group φ we have $f_0(B_{ik}) = f_0$, $f_{ik} = f_0 + \bar{f}_{ik}$, where $\bar{f}_{ik} = \bar{f}(B_{ik})$. Next one can choose $\varepsilon > 0$ such that, for all $|\varepsilon_{lm}| < \varepsilon$ ($l = 1, 2, 3$; $m = 1, 2, 3$), one has $\Delta_{ik} = \bar{f}_{ik}/f_0 < 1$; therefore, taking $1/f_0^m$ outside the parentheses and expanding $\Delta_i\Sigma$ by Newton's binomial formula, we obtain the equality:

$$\Delta_i \Sigma = \sum_{x,y,z} \frac{1}{f_0^m} \left[-m(\Delta_{i1} + \Delta_{i2} + \dots + \Delta_{ili}) + \frac{m(m+1)}{2} (\Delta_{i1}^2 + \Delta_{i2}^2 + \dots + \Delta_{ili}^2) \right] + R(\varepsilon_{ik}^3). \quad (2)$$

As a result of the computations we obtain

$$\Delta_{i1} + \Delta_{i2} + \dots + \Delta_{ili} = \frac{1}{6l_i} (3\varepsilon_{11} + 3\varepsilon_{22} + 3\varepsilon_{33} - \varepsilon_{12} - \varepsilon_{13} - \varepsilon_{23}), \quad (3)$$

$$\Delta_{i1}^2 + \Delta_{i2}^2 + \dots + \Delta_{ili}^2 = \frac{1}{6l_i} [\beta + 2(\bar{\beta} - 2\beta)xyz(x+y+z)/f_0^2], \quad (4)$$

where

$$\beta = 3\varepsilon_{11}^2 + 3\varepsilon_{22}^2 + 3\varepsilon_{33}^2 + \varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2 + 2\varepsilon_{11}\varepsilon_{22} + 2\varepsilon_{11}\varepsilon_{33} + 2\varepsilon_{22}\varepsilon_{33} - 2\varepsilon_{11}\varepsilon_{12} - 2\varepsilon_{11}\varepsilon_{13} - 2\varepsilon_{22}\varepsilon_{12} - 2\varepsilon_{22}\varepsilon_{23} - 2\varepsilon_{33}\varepsilon_{13} - 2\varepsilon_{33}\varepsilon_{23}$$

$$\bar{\beta} = 9\varepsilon_{11}^2 + 9\varepsilon_{22}^2 + 9\varepsilon_{33}^2 + \varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2 + 2\varepsilon_{11}\varepsilon_{22} + 2\varepsilon_{11}\varepsilon_{33} + 2\varepsilon_{22}\varepsilon_{33} - 6\varepsilon_{11}\varepsilon_{12} - 6\varepsilon_{11}\varepsilon_{13} - 6\varepsilon_{22}\varepsilon_{12} - 6\varepsilon_{22}\varepsilon_{23} - 6\varepsilon_{33}\varepsilon_{13} - 6\varepsilon_{33}\varepsilon_{23}$$

β is the sum of Δ_{ik}^2 at the points representing the minimum of the form f_0 ; β is positive definite, $\bar{\beta}$ is an indefinite form. From the equation $D = D_0 = \frac{1}{2} (D$ and D_0 are the determinants of the forms f and f_0) we find, restricting ourselves only to the quadratic part:

$$3\varepsilon_{11} + 3\varepsilon_{22} + 3\varepsilon_{33} - \varepsilon_{12} - \varepsilon_{13} - \varepsilon_{23} = \varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2 - 4\varepsilon_{11}\varepsilon_{22} - 4\varepsilon_{11}\varepsilon_{33} - 4\varepsilon_{22}\varepsilon_{33} + 2\varepsilon_{11}\varepsilon_{23} + 2\varepsilon_{22}\varepsilon_{13} + 2\varepsilon_{33}\varepsilon_{12} - \varepsilon_{12}\varepsilon_{13} - \varepsilon_{13}\varepsilon_{23} - \varepsilon_{23}\varepsilon_{12} \quad (5)$$

Substituting the values (3) and (4) into (2), and then (2) into (1), and denoting

$$4 \sum_{x \neq y \neq z} \frac{1}{f_0^m} + 2 \sum_{x=y, z} \frac{1}{f_0^m} + 2 \sum_{x, y=z} \frac{1}{f_0^m} + \frac{2}{3} \sum_{x=y=z} \frac{1}{f_0^m} + \frac{1}{2} \sum_{x=y=-z} \frac{1}{f_0^m} + 2 \sum_{x=y, 0} \frac{1}{f_0^m} + \sum_{x, 0, 0} \frac{1}{f_0^m} = \theta_m,$$

and by S_m the same sum, but with the quantity $xyz(x+y+z)/f_0^{m+2}$ instead of $1/f_0^m$, we obtain

$$\Delta \Sigma = -m \mathcal{L} \theta_m + \frac{1}{2} m(m+1) \beta \theta_m + m(m+1) (\bar{\beta} - 2\beta) S_m. \quad (6)$$

$\Delta \Sigma$, in view of (5), is a completely determined quadratic form in the variables ε_{ik} and the parameters θ_m and S_m . Let us investigate this form.

$m \geq 4$. In view of (4), the inequality holds:

$$-\mathcal{L}\theta_m + \frac{1}{2}(m+1)\beta\theta_m + (m+1)(\bar{\beta} - 2\beta)S_m > -\mathcal{L}\theta_m + \frac{1}{2}(m+1)\beta. \quad (7)$$

In order not to interrupt the sequence of the exposition, the estimates of θ_m and S_m will be given later; for now we shall use the ready result

$$1 < \theta_4 < 1.2. \quad (8)$$

For $m \geq 4$ and condition (8), the right-hand side of inequality (7) will always be positive definite, while for $m = 2, 3$ it will already be an indefinite form. For $m \geq 4$ the theorem is proved.

For $m = 2, 3$ the estimates hold

$$2.109 < \theta_2 < 2.3, \quad -0.111 < S_2 < -0.067; \quad (9)$$

$$1.14 < \theta_3 < 1.4, \quad -0.04 < S_3 < 0. \quad (10)$$

The variations ε_{ik} are not arbitrary, but satisfy the equation $D = 1/2$, from which we find

$$\varepsilon_{11} = (\varepsilon_{12} + \varepsilon_{13} + \varepsilon_{23} - 3\varepsilon_{22} - 3\varepsilon_{33})/3 \quad (11)$$

(we take into account only the linear part, since the quadratic part, when substituted into (6), already gives the fourth degree in ε_{ik}). Substituting expression (11) into (6), we obtain

$$\begin{aligned} \Delta\Sigma = & 6[(m-1)\theta_m + 4(m+1)S_m](\varepsilon_{22}^2 + \varepsilon_{33}^2) + [(m-2)\theta_m - \\ & - 4(m+1)S_m](\varepsilon_{12}^2 + \varepsilon_{13}^2) + (2m-3)\theta_m\varepsilon_{23}^2 + 6[(m-1)\theta_m + \\ & + 4(m+1)S_m]\varepsilon_{22}\varepsilon_{33} - 2[(m-1)\theta_m + 4(m+1)S_m](\varepsilon_{22}\varepsilon_{12} + \varepsilon_{33}\varepsilon_{13}) + \\ & + [(m-1)\theta_m + 4(m+1)S_m](\varepsilon_{22}\varepsilon_{13} + \varepsilon_{33}\varepsilon_{12}) - 5[(m-1)\theta_m + \\ & + 4(m+1)S_m](\varepsilon_{22}\varepsilon_{23} + \varepsilon_{33}\varepsilon_{23}) + [(2-m)\theta_m + 4(m+1)S_m]\varepsilon_{12}\varepsilon_{13} + \\ & + [\theta_m + 8(m+1)S_m](\varepsilon_{12}\varepsilon_{23} + \varepsilon_{13}\varepsilon_{23}) \end{aligned}$$

a quadratic form in five variables.

The principal minors of the determinant of this form will be

$$D_1 = C_1[(m-1)\theta_m + 4(m+1)S_m],$$

$$D_2 = C_2[(m-1)\theta_m + 4(m+1)S_m]^2,$$

$$D_3 = C_3(11m-29) \left[\theta_m + \frac{100(m+1)}{(29-11m)} S_m \right],$$

$$D_4 = C_4 \left[\frac{3}{2}(m-3)\theta_m - 18(m+1)S_m \right] [(17m-35)\theta_m - 76(m+1)S_m],$$

$$D_5 = C_5 \left[\frac{3}{2}(m-3)\theta_m - 18(m+1)S_m \right] [(m^2 - 4m + 3)\theta_m^2 - 8m(m+1)\theta_m S_m - 48(m+1)^2 S_m^2],$$

where $C_i > 0$ ($i = 1, 2, 3, 4, 5$) are constants.

It is easy to verify for $m = 2, 3$ and under conditions (9) and (10) that the principal minors are positive; consequently, $\Delta\Sigma$ is a positive definite form, which proves the theorem. From this there also follows once more the proof for $m \geq 4$, since for $\theta_m > 1$ and $S_m < 0$ the D_i ($i = 1, 2, 3, 4, 5$) are positive.

Estimations of θ_m and S_m . The equalities are obvious

$$\sum_{x=y=z} \frac{1}{f_0^m} = \frac{1}{6^m} \zeta(2m), \quad \sum_{x=y=-z} \frac{1}{f_0^m} = \frac{1}{2^m} \zeta(2m), \quad (12)$$

$$\sum_{x=y, 0} \frac{1}{f_0^m} = \frac{1}{3^m} \zeta(2m), \quad \sum_{x, 0, 0} \frac{1}{f_0^m} = \zeta(2m).$$

Let the sums

$$\sum_{x \neq y \neq z} \frac{1}{f_0^m}, \quad \sum_{x=y, z} \frac{1}{f_0^m}, \quad \sum_{x, y=z} \frac{1}{f_0^m}, \quad (13)$$

$$\sum_{x=-y=z} \frac{xyz(x+y+z)}{f_0^{m+2}}, \quad \sum_{x=-y, z} \frac{xyz(x+y+z)}{f_0^{m+2}}, \quad \sum \frac{xyz(x+y+z)}{f_0^{m+2}} \quad (14)$$

be computed for all points for which $f_0 \leq c^2$. The sums of the series (13) and (14) for which $f_0 > c^2$ will be estimated with the aid of an integral.

In the space E^4 , in Cartesian coordinates, consider the surface $t = 1/f_0^m$. In the space E^3 the set K forms a cubic lattice Γ . The region enclosed between Φ and the corresponding part of the surface is filled by prisms whose bases are

the cubes of the lattice Γ , and whose height is the smallest value taken by the function $1/f_0^m$ in the given cube. The step body thus obtained lies beneath the surface. To each point $B_1(x, y, z)$ of the lattice Γ we assign the cube Π with vertices $B_1(x, y, z)$, $B_2(x-1, y, z)$, $B_3(x, y-1, z)$, $B_4(x-1, y-1, z)$, $B_5(x, y, z-1)$, $B_6(x-1, y, z-1)$, $B_7(x, y-1, z-1)$, $B_8(x-1, y-1, z-1)$.

Under this correspondence: 1) the value $1/f_0^m$ will be the smallest in the cube Π ; 2) for all points $B_1(x, y, z) \in \Phi$, with the exception of the plane $y+z=1$, the cubes Π fall completely into the region $\bar{\Phi}$. We reason similarly for points lying on the boundary of Φ (here, instead of cubes there will be squares). Comparing the volumes of the step body in the region lying under the surface $1/f_0^m$, and taking into account that

$$1/f_0^m(x, y, -y+1) < 1/f_0^m(x, y, -y),$$

we obtain, for $f_0 \geq c^2$, the estimates

$$\sum_{x \neq y \neq z} \frac{1}{f_0^m} < \frac{\pi\sqrt{2}}{12(2m-3)(c-\sqrt{6})^{2m-3}} + \frac{\pi}{2(2m-2)(c-\sqrt{2})^{2m-2}}, \quad (15)$$

$$\sum_{x=y, z} \frac{1}{f_0^m} < \frac{14\pi}{45\sqrt{2}(2m-2)(c-\sqrt{6})^{2m-2}}, \quad \sum_{x, y=z} \frac{1}{f_0^m} < \frac{\pi}{5\sqrt{2}(2m-2)(c-\sqrt{6})^{2m-2}}.$$

The following inequalities hold:

$$\begin{aligned} xyz(x+y+z) &< \frac{1}{3}f_0^2, \\ xyz(x+y+z) &\leq \frac{1}{7}f_0^2, \quad \text{if } z > 0, \\ xyz(x+y+z) &\leq \frac{1}{12}f_0^2, \quad \text{if } y = z. \end{aligned} \quad (16)$$

The values of the sums (13) and (14) for $m=2$ and $f_0 \leq 225$ were computed on a computer by L. Voitishchek of Novosibirsk; they are respectively equal to

$$0.148072 \dots, \quad 0.028883 \dots, \quad 0.017752 \dots, \quad -0.012944 \dots, \quad -0.003021 \dots, \quad 0.000978 \dots \quad (17)$$

Taking into account the values (17) and computing (12) and (15), we obtain the estimate (9) for θ_2 . Using the values (17) and the inequalities (16), and then the estimates (15), we obtain an estimate for S_2 . Knowing the estimates for θ_2 and S_2 , we estimate θ_3 and S_3 .

As a result of computing (12) we obtain the lower estimate (10) for θ_3 . Next we note that, among all points for which the values $1/f_0^m$ do not enter into (12), the point $B_1(2, 1, -1)$ gives the smallest value $f_0 = 5$; hence we easily obtain the upper estimate (10) for θ_3 . The estimates for S_3 and θ_4 are computed analogously.

In conclusion I express my gratitude to L. Voitishchek for computing the sums.

Mathematical Institute named after V. A. Steklov
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1. B. N. Delone, N. N. Sandakova, S. S. Ryshkov, DAN, **162**, No. 6 (1965)

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