

SYMMETRIZABLE SPACES

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Abstract

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MATHEMATICS

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SYMMETRIZABLE SPACES AND FINAL COMPACTNESS

(Presented by Academician P. S. Aleksandrov, 4 X 1966)

All the proofs in this article are based on Theorem 1; its formulation is based on the notion, recently proposed by A. V. Arhangel'skii, of a weakly discrete system of sets. The proof of Theorem 1 rests on a technique developed by Bing for the case of spaces with a refining sequence of covers and subsequently applied by MacAuley in the study of strongly symmetrizable spaces.

The proof of Theorem 2 is based on the main idea of the proof of the theorem on the metrizability of a symmetrizable bicomactum, due to A. V. Arhangel'skii and V. Stoyanovsky (see ⁽¹⁾). Theorem 2 may be regarded as a partial answer to Michael's question: is every closed set in a symmetrizable space a set of type G_δ .

Definition 1 (A. Arhangel'skii). A system $\{M_\alpha\}_{\alpha \in A}$ of subsets of a topological space X is called **weakly discrete** if from $x_\alpha \in M_\alpha$ it follows that the system of points $\{x_\alpha\}_{\alpha \in A}$ is discrete in X .

Theorem 1. *In every open cover of a symmetrizable space one can inscribe a σ -weakly discrete cover.*

Proof. Let X be a space with symmetrix d . We shall call a system of subsets $\{M_\alpha\}_{\alpha \in A}$ **discrete with respect to the symmetrix d** if for every $x \in X$ there exists $\varepsilon = \varepsilon(x) > 0$ such that $O_\varepsilon x$ meets no more than one element of the system $\{M_\alpha\}_{\alpha \in A}$.

Lemma. *If the system $\{M_\alpha\}_{\alpha \in A}$ is discrete with respect to the symmetrix d and, for every $\alpha \in A$, M_α is closed in X , then the system $\{M_\alpha\}_{\alpha \in A}$ is discrete in X .*

Indeed, if for some $A' \subset A$ the set $\bigcup_{\alpha \in A'} M_\alpha$ is not closed in X , then there exists a point

$$x \in X \setminus \bigcup_{\alpha \in A'} M_\alpha$$

such that

$$d\left(x, \bigcup_{\alpha \in A'} M_\alpha\right) = 0.$$

But the system $\{M_\alpha\}_{\alpha \in A}$ is discrete with respect to the symmetrix d ; consequently,

$$d(x, M_{\alpha_0}) = 0$$

for some $\alpha_0 \in A'$. Hence $x \in M_{\alpha_0}$, and this contradicts the condition

$$x \in X \setminus \bigcup_{\alpha \in A'} M_\alpha.$$

Let now $\omega = \{U_\alpha\}_{\alpha \in A}$ be an open cover of the space X , with A a well-ordered set. For each natural n and each $\alpha \in A$, put

$$M_\alpha^n = \mathcal{E}\{x \in X, x \in U_\alpha \setminus \bigcup_{\beta < \alpha} U_\beta, O_{1/n}x \subset U_\alpha\}.$$

We shall show that, for each fixed n , the system $\{M_\alpha^n\}_{\alpha \in A}$ is discrete with respect to the symmetrix d . Indeed, let $x \in X$. Then, for some $\alpha \in A$,

$$x \in U_\alpha \setminus \bigcup_{\beta < \alpha} U_\beta$$

and, for some $m \geq n$,

$$O_{1/m}x \subset U_\alpha$$

(whence it is clear that $\{M_\alpha^n\}_{\alpha \in A, n=1,2,\dots}$ is a cover). Suppose that for some β ,

$$O_{1/m}x \cap M_\beta^n \neq \Lambda.$$

This is impossible for $\beta > \alpha$, since in that case

$$M_\beta^n \subset U_\beta \setminus U_\alpha,$$

whereas

$$O_{1/m}x \subset U_\alpha.$$

But $O_{1/m}x \cap M_\beta^n \neq \Lambda$ also cannot occur for $\beta < \alpha$, since then, from

$$y \in O_{1/m}x \cap M_\beta^n,$$

it follows that

$$x \in O_{1/n}y \subset O_{1/m}y \subset U_\beta,$$

which contradicts the relation

$$x \in U_\alpha \setminus \bigcup_{\gamma < \alpha} U_\gamma \subset U_\alpha \setminus U_\beta.$$

To complete the proof of Theorem 1 it remains only to apply the lemma.

Theorem 2. *A finally compact symmetrizable space is hereditarily finally compact.*

Proof. Let X be a Hausdorff finally compact space with a symmetric d , and let F be a closed subspace of the space X . Then $X' = X \setminus F$ is a symmetrizable space, as an open subspace of a symmetrizable space. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of the space, where A is a well-ordered set. Put again

$$M_\alpha^n = \mathcal{E}\{x \in X', x \in \overline{U}_\alpha \setminus \bigcup_{\beta < \alpha} U_\beta, O_{1/n}x \subset U_\alpha\}$$

and suppose that for some n the set of nonempty elements of the system $\{M_\alpha^n\}_{\alpha \in A}$ is uncountable. Choose from each nonempty element of the system $\{M_\alpha^n\}_{\alpha \in A}$ a point $x_\alpha \in M_\alpha^n$.

Since

$$F = \bigcap_{m=1}^{\infty} O_{1/m}F,$$

there exist $\varepsilon > 0$ and $A' \subset A$ such that:

- 1) $x_\alpha \notin O_\varepsilon F$ for $\alpha \in A'$, and 2) the system $\{x_\alpha\}_{\alpha \in A'}$ is uncountable.

The set $\mathcal{E}\{x_\alpha : \alpha \in A'\}$ is not closed in X , because, as we saw in the proof of Theorem 1, it is discrete in the space X' , and it has limit points by virtue of the final compactness of the space X and the large cardinality of the set $\mathcal{E}\{x_\alpha : \alpha \in A'\}$. Consequently, there exists a point $x_0 \in X \setminus \mathcal{E}\{x_\alpha : \alpha \in A'\}$ such that $d(x_0, \mathcal{E}\{x_\alpha : \alpha \in A'\}) = 0$.

From the last equality and the definition of the set A' it follows that $x_0 \in X'$, but this contradicts the discreteness of the system $\{x_\alpha\}_{\alpha \in A'}$ in X . The contradiction completes the proof of Theorem 2, since it shows that from the cover $\{M_\alpha^n\}_{\alpha \in A, n=1,2,\dots}$, and consequently also from the cover $\{U_\alpha\}_{\alpha \in A}$, one can choose a countable subcover.

Theorem 3. *A symmetrizable space is finally compact if and only if every discrete system of points in it is countable.*

Proof follows directly from Theorem 1, the lemma, and the known fact that in any finally compact space every discrete system is countable.

Corollary 1. *A Hausdorff symmetrizable hereditarily separable space is hereditarily finally compact*.*

Corollary 2. *A Hausdorff countably compact symmetrizable space is compact**.*

Proof. If X is a countably compact space, then in X every discrete system of points is finite. Hence, if X is a symmetrizable space, then by Theorem 3 it follows that X is finally compact and, consequently, bicomact. By the main

theorem of Arhangel'skii and Stoyanovskij formulated at the beginning ⁽¹⁾, *** a symmetrizable bicomactum is compact.

Theorem 4. *A symmetrizable M -space (in the sense of Morita ⁽⁶⁾) is metrizable.*

Proof. By Morita's theorem (⁽⁶⁾, Theorem 5.3), every M -space admits a closed mapping onto a metric space under which the preimage of each point is a countably compact space. Hence, from Corollary 2 we obtain that every symmetrizable M -space is a feathered paracompactum (see ⁽³⁾), and an arbitrary feathered paracompactum with a symmetric is metrizable, by a theorem of A. V. Arhangel'skii.

* If the word symmetrizable is replaced by strongly symmetrizable, one obtains a statement proved by Sider ⁽⁷⁾. His arguments do not carry over to the case of symmetrizable spaces.

** If the word symmetrizable is replaced by strongly symmetrizable, one obtains a statement proved by Nemytskii ⁽⁵⁾. His arguments do not carry over to the case of symmetrizable spaces.

***) The key role in the proof of this theorem is played by the special case of Theorem 2 concerning bicompacta. Then a known theorem of Nemytskii is applied ⁽⁵⁾.

Definition 2 (A. Arhangel'skii (2)). A symmetric d of a symmetrizable space X is called a **symmetric satisfying the weak Cauchy condition*** if, from $A \subset X$ and $[A] \neq A$, it follows that for every $\varepsilon > 0$ there are distinct points x and y in A for which $d(x, y) < \varepsilon$.

In Arhangel'skii's paper (2) the following assertion is proved: a factor, weakly uniformly** image of a metric space is symmetrizable with a symmetric satisfying the weak Cauchy condition.

Using the ideas of (2), we prove the following theorem.

Theorem 5. *If the Hausdorff symmetric d of a finally compact symmetrizable space X satisfies the weak Cauchy condition, then for every subspace X' of the space X and every $\varepsilon > 0$, from the cover $\{O_\varepsilon x\}_{x \in X'}$ of the space X' one can choose a countable cover of the space X' .*

Proof. Starting with an arbitrary point $x_0 \in X'$, we construct a system of points $\{x_\alpha \in X'\}_{\alpha \in A}$ such that: 1) A is well ordered; 2)

$$X' = \bigcup_{\alpha \in A} O_\varepsilon x_\alpha;$$

3)

$$x_\alpha \in X' \setminus \bigcup_{\beta < \alpha} O_\varepsilon x_\beta.$$

The last condition can obviously be replaced by 3') for $\alpha \neq \beta$, $d(x_\alpha, x_\beta) \geq \varepsilon$. The system of points $\{x_\alpha\}_{\alpha \in A}$ so constructed turns out to be discrete in X . Indeed, if we suppose that for some subset A' of the set A the set $\mathcal{E}\{x_\alpha : \alpha \in A'\}$ is not closed in X , then we would obtain that for some $\alpha \neq \beta$, $d(x_\alpha, x_\beta) < \varepsilon$, which contradicts condition 3'). Consequently, the system of points $\{x_\alpha\}_{\alpha \in A}$ is countable, as is every discrete system of points in a finally compact space. The theorem is proved.

Corollary 3. *A symmetrizable finally compact space, some symmetric of which satisfies the weak Cauchy condition, is hereditarily separable.*

Proof. Let a symmetric d of the finally compact symmetrizable space X satisfy the weak Cauchy condition, and let X' be an arbitrary subspace of the space X . By Theorem 5, for every natural n there exists a sequence of points $\{x_m^n\}_{m=1}^\infty$ such that: 1) $x_m^n \in X'$ for every m , and 2)

$$X' = \bigcup_{m=1}^\infty O_{1/n} x_m^n.$$

It is easy to verify that the countable set $\mathcal{E}\{x_m^n, n, m = 1, 2, \dots\}$ is everywhere dense in X' .

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REFERENCES

1. A. V. Arhangel'skii, UMN, 21, no. 4 (130), 133 (1966).
2. A. V. Arhangel'skii, DAN, 164, no. 2 (1965).
3. A. V. Arhangel'skii, Matem. sborn., 67 (109), no. 1 (1965).
4. P. S. Aleksandrov, V. V. Nemytskii, Matem. sborn., 3, 3, 665 (1938).
5. V. Niemytzki, Math. Ann., 101, no. 5, 666 (1931).
6. K. Morita, Proc. Japan Acad., 39, 150 (1963).

7. J. G. Geder, Pacific J. Math., 11, no. 1 (1961).

* A symmetric d is called a **Cauchy symmetric** (see (4)) if, from

$$\lim_{n \rightarrow \infty} d(x_n, x) = 0,$$

it follows that

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0.$$

** A mapping $f : X \rightarrow Y$ of a symmetrizable space X onto a topological space Y is called **weakly uniformly continuous** if for any point $y \in Y$ and its neighborhood U ,

$$\rho(f^{-1}y, X \setminus f^{-1}U) > 0,$$

where ρ is a fixed symmetric of the space X .

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