

ON FACTOR REPRESENTATIONS OF TYPE $\backslash(\mathrm{II})_1\backslash$

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Abstract

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MATHEMATICS

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ON FACTOR REPRESENTATIONS OF TYPE Π_1 FOR THE CLIFFORD ALGEBRA

(Presented by Academician I. M. Vinogradov on VI 6, 1966)

1. A representation of the Clifford (or spinor) algebra with an infinite number of generators is a set of linear self-adjoint operators $\{A_k\}_1^\infty$, acting in a separable Hilbert space H and satisfying the relations

$$A_k^2 = 1, \quad A_k A_l + A_l A_k = 0 \quad (k \neq l). \quad (1)$$

If the weakly closed ring M , generated by the operators $\{A_k\}_1^\infty$, is a factor in the sense of Murray and von Neumann, then such a representation will be called a **factor representation**.

Some properties of factor representations were noted in ⁽⁶⁾. In the present note we give a complete classification of factor representations of type Π_1 , and also consider the question of decomposing a factor representation of type Π_1 into irreducible representations.

2. Let $\{A_k\}_1^\infty$ and $\{B_k\}_1^\infty$ be two factor representations acting in the spaces H_1 and H_2 ; denote the corresponding factors by M_1 and M_2 .

Definition. The factor representations $\{A_k\}_1^\infty$ and $\{B_k\}_1^\infty$ will be called **algebraically isomorphic** if one can establish an algebraic isomorphism φ between the factors M_1 and M_2 in such a way that $\varphi(A_k) = B_k$ ($k = 1, 2, \dots$).

It can be shown that there exist algebraically nonisomorphic factor representations of type III and algebraically nonisomorphic representations of type Π_∞ .

Theorem 1. *All factor representations of the Clifford algebra of type Π_1 are algebraically isomorphic.*

We outline the proof of the theorem. Introduce in the ring M_i ($i = 1, 2$) a scalar product by putting

$$\langle A, B \rangle = T_i(AB^*),$$

where T_i is the relative trace for the factor M_i , and $A, B \in M_i$. Then M_i becomes a pre-Hilbert space. We denote by $Q(M_i)$ the completion of M_i with respect to this scalar product. We shall assume that $T_i(I_i) = 1$, where I_i is the identity operator. Consider monomials of the form

$$A_z = A_{s_1} A_{s_2} \cdots A_{s_n},$$

where z denotes a set of indices $s_1 < s_2 < \cdots < s_n$; $n = 1, 2, \dots$; $s_i = 1, 2, \dots$ ($i = 1, 2, \dots, n$). It is not difficult to verify that

$$T_1(A_z) = 0$$

for all z . Consequently, the monomials of the form A_z and the identity operator I_1 form a complete orthonormal basis in the space $Q(M_1)$. Every element D of $Q(M_1)$ can be expanded in a series with respect to this basis,

$$D \sim \sum_z d_z A_z \quad \left(\sum_z |d_z|^2 < \infty \right).$$

Now establish a correspondence between $Q(M_1)$ and $Q(M_2)$ by the rule

$$\varphi(I_1) = I_2; \quad \varphi(A_z) = B_z.$$

It is not difficult to verify that $\varphi(C + D) = \varphi(C) + \varphi(D)$ and $\varphi(C^*) = \varphi(C)^*$, and if C and D belong to M_1 (i.e., are bounded operators), then

$$\varphi(CD) = \varphi(C)\varphi(D).$$

Consequently, if C is a positive definite operator from M_1 , then $\varphi(C)$ is also a positive definite operator. But then every operator D from M_1 , under the mapping φ , falls into M_2 , since the norm of the operator $\|D\|$ is the least number $a \geq 0$ for which the operator $a^2 I_1 - D^* D$ is positive definite. From the considerations given, it follows that $\varphi(M_1) \subset M_2$; similarly we obtain that $\varphi^{-1}(M_2) \subset M_1$. Hence, $\varphi(M_1) = M_2$.

In ⁽¹⁾ von Neumann constructed an example of a factor-representation of type Π_1 . From Theorem 1 it follows that all factor-representations of type Π_1 for the Clifford algebra are algebraically isomorphic to this representation.

3. Let us turn to the question of the unitary classification of factor-representations of type Π_1 .

Let M be a factor of type II_1 , whose elements act in the space H . By M' , as usual, denote the commutant of the factor M . Let $f \in H$; then by $H_f^{M'}$ denote the closure of the set $M'f$ with respect to the norm of H , and by H_f^M the closure of Mf . If P_1^f is the projector onto $H_f^{M'}$, and P_2^f the projector onto H_f^M , then

$$T_M(P_1^f) = cT_{M'}(P_2^f),$$

where the constant c does not depend on f , and T_M and $T_{M'}$ are the traces for M and M' , respectively. Introduce for consideration the number ⁽²⁾

$$\theta = T_{M'}(I)/cT_M(I). \quad (2)$$

Under the assumption that M is a finite factor, θ varies in the range $0 < \theta \leq \infty$.

Theorem 2. *For every number θ ($0 < \theta \leq \infty$) there exists a factor-representation of the Clifford algebra of type II_1 , and this number θ determines the factor-representation uniquely up to unitary equivalence.*

For other representations of the Clifford algebra with a continuous ergodic measure ⁽³⁾, such a simple classification is unknown.

4. Analyzing the example of von Neumann, which was already mentioned, one can easily verify that the factor-representation of the Clifford algebra of type II_1 decomposes into a direct integral of irreducible representations for which $\nu = 1$, according to the classification of Gårding and Wightman ⁽³⁾. It is of interest to determine whether irreducible representations for which $\nu \neq 1$ will enter into the decomposition of a factor-representation of type II_1 .

Theorem 3. *A factor-representation of type II_1 for the Clifford algebra can be decomposed into a direct integral of irreducible representations for which $\nu = 2^n$, where $n = 0, 1, 2, \dots$*

The proof of the theorem consists in constructing, for $\nu = 2^n$ ($n = 0, 1, \dots$), a factor-representation of type II_1 for the Clifford algebra that has the properties of interest to us. We note that the very method of constructing factors of type II_1 was previously unknown.

Let Γ be the space of sequences $x = x_1 x_2 \dots$, where $x_i = 0$ or 1 . One may regard Γ as a group if componentwise addition modulo 2 is introduced for sequences. By δ_k ($k = 1, 2, \dots$) denote the sequence from Γ with a one in the k -th place and zeros in all the others. The Haar measure μ of the group Γ coincides with Lebesgue measure under the natural mapping of Γ onto the unit interval.

Let H be the Hilbert space of vector-functions $f(x, y)$ on $\Gamma \times \Gamma$, whose values belong to a 2^n -dimensional complex pro-

space R_n . We define the scalar product in $\tilde{H}$ by the formula

$$\langle f, g \rangle = \int_{\Gamma \times \Gamma} (f(x, y), g(x, y)) d\mu(x) d\mu(y),$$

where $(\cdot, \cdot)$ is the scalar product in R_n .

We define a representation of the anticommutation relations $\{A_k, B_k\}_1^\infty$ in $\tilde{H}$ as follows:

$$\begin{aligned} A_k f(x, y) &= j_k(x) c_k(x + y) f(x + \delta_k, y), \\ B_k f(x, y) &= i^{-1} j_{k+1}(x) c_k(x + y) f(x + \delta_k, y) \quad (k = 1, 2, \dots), \end{aligned} \quad (3)$$

where $j_k(x) = (-1)^{x_1 + \dots + x_{k-1}}$, and $\{c_k(x)\}_1^\infty$ are measurable functions on Γ , whose range consists of unitary operators in the space R_n . Similarly we define the representation $\{\tilde{A}_l, \tilde{B}_l\}_1^\infty$:

$$\begin{aligned} \tilde{A}_l f(x, y) &= j_l(y) c_l(x + y) f(x, y + \delta_l), \\ \tilde{B}_l f(x, y) &= i^{-1} j_l(y) c_l(x + y) f(x, y + \delta_l) \quad (l = 1, 2, \dots). \end{aligned} \quad (3')$$

Before explicitly writing the expression for $c_k(x)$ ($k = 1, 2, \dots$), we make a remark. Denote by $\{p_s\}_1^{2n}$ an irreducible representation of the Clifford algebra in R_n with $2n$ generators $p_k = p_k^*$, $p_k^2 = 1$ ($k = 1, 2, \dots, n$), $p_k p_l + p_l p_k = 0$ ($l \neq k$)⁴. Now put

$$c_{2nk+r}(x) = s_{2nk+r}(x) p_r, \quad (4)$$

where $k = 0, 1, 2, \dots$; $0 \leq r \leq 2n$,

$$s_{2nk+r}(x) = (-1)^{\sum_{i=1}^{2nk+r} x_i - \sum_{j=0}^k x_{2nj+r}}.$$

It is not hard to verify that the relations

$$\begin{aligned} c_k(x + \delta_k) &= c_k^*(x), \\ c_k(x) c_l(x + \delta_k) &= c_l(x) c_k(x + \delta_l) \quad (l \neq k) \end{aligned} \quad (5)$$

are satisfied.

From the results of (3) it follows that, for almost every y , $\{A_k, B_k\}_1^\infty$ in (3) define an irreducible representation, and for almost every x , $\{\tilde{A}_k, \tilde{B}_k\}_1^\infty$ also define an irreducible representation. On the basis of Theorem 3⁶ we assert that the weakly closed ring M generated by the operators $\{A_k, B_k\}_1^\infty$, and also

the ring $\tilde{M}$ generated by $\{\tilde{A}_k, \tilde{B}_k\}_1^\infty$, are factors. Moreover, it turns out that $\tilde{M} = M'$ and $M = \tilde{M}'$.

Let us verify that M and $\tilde{M}$ are factors of type II_1 . For this it is enough to show⁵ that on all operators from M one can define a linear homogeneous positive functional T , satisfying the conditions: 1) $T(I) = 1$, where I is the identity operator; 2) $T(AB) = T(BA)$ ($A, B \in M$); 3) if P is a projector from M and $T(P) = 0$, then $P = 0$.

Let $e_1, \dots, e_{2^n}$ be an orthonormal basis in R_n , whose elements permute the operators $p_1, \dots, p_{2^n}$. Denote $\varphi_0(x, y) \equiv 1$ and consider the vector-function $f(x, y) = \varphi_0(x, y)e_1$. Then the functional $T(A) = \langle Af, f \rangle$, where $A \in M$, has all the properties listed.

We note that if in our construction $n = 0$, then we obtain an example of von Neumann¹. The case $n \neq 0$ has not been considered previously.

It seems interesting that if, as the measure $\mu(y)$, one takes an arbitrary measure not equivalent to Lebesgue measure, but quasi-invariant and ergodic with respect to shifts by δ_k ($k = 1, 2, \dots$), then we obtain a factor-representation of type III for the Clifford algebra. For the case $n \neq 0$ these representations, apparently, were not known.

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