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Abstract

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MATHEMATICS

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PARABOLIC EQUATIONS IN SPACES OF MEANS

(Presented by Academician I. G. Petrovskii, 25 I 1967)

1. In the present paper we continue the study, begun in ⁽¹⁻⁴⁾, of the coercivity problem for the Cauchy problem

$$v'(t) + Av(t) = f(t) \quad (0 \leq t \leq 1), \quad v(0) = v_0 \quad (1)$$

in a Banach space E . This problem is studied in new series of spaces—integral analogues of abstract Hölder spaces and abstract spaces of the type of the spaces of S. L. Sobolev—L. N. Slobodetskii. Here, just as in Hölder spaces (see ⁽¹⁾), the necessary and sufficient condition for coercivity is the strong positivity in E of the operator A . At the same time the construction of these new spaces is close to the construction of Bochner spaces, for which the coercivity problem has not yet been solved in such generality.

The coercivity problem is also solved in spaces of generalized functions whose elements are derivatives, in the sense of S. L. Sobolev—L. Schwartz, of elements of the spaces indicated above. Spaces of generalized functions may be obtained as completions of the set of smooth functions in the norms of their antiderivatives. This gave grounds for calling the spaces of generalized functions, as well as the spaces of their antiderivatives, spaces of means.

2. Let $F(t)$ be a continuous function on $[0, 1]$ with values in E ;

$$\omega(t, F) = \max_{0 \leq |t_1 - t_2| \leq t} \|F(t_1) - F(t_2)\|_E$$

is its adjusted modulus of continuity; $\varphi(t, F) = \max_{t \leq s \leq 1} \omega(s, F)/s$ is a monotonically decreasing majorant of the function $\omega(t, F)/t$. We shall say that the function $F(t)$ belongs to the set $S_p(E)$ ($1 \leq p \leq \infty$), if

$$\|F(t)\|_{S_p} \equiv \max_{0 \leq t \leq 1} \|F(t)\|_E + \left[\int_0^1 \varphi^p(t, F) dt \right]^{1/p} < \infty. \quad (2)$$

The set $S_p(E)$ is a Banach space with norm $\|E(t)\|_{S_p}$. The totality of functions $F(t) \in S_p(E)$ satisfying the condition $F(0) = 0$ forms the subspace $S_p^0(E)$.

Let $C_\alpha(E)$ ($0 < \alpha \leq 1$) be the space of functions defined on $[0, 1]$ with values in E satisfying the Hölder condition with exponent α ; $C_0(E)$ is the space of continuous functions.

Lemma 1. If $F(t) \in S_p(E)$ ($1 \leq p < \infty$), then $F(t) \in C_{1/q}(E)$ ($1/p + 1/q = 1$), and $\|F(t)\|_{C_{1/q}} \leq K\|F(t)\|_{S_p}$. If, however, $F(t) \in C_{1/q+\varepsilon}(E)$ ($\varepsilon > 0$), then $F(t) \in S_p(E)$, and

$$\|F(t)\|_{S_p} \leq K\varepsilon^{-1}\|F(t)\|_{C_{1/q+\varepsilon}}. \quad (3)$$

The spaces S_∞ and C_1 coincide.

Let E_1 be a dense subset of E which is a Banach space and $\|v\|_E \leq K\|v\|_{E_1}$ for every $v \in E_1$. By $C_\infty(E_1)$ we denote the set of infinitely differentiable functions defined on $[0, 1]$ with values in E_1 . By $C_\infty^0(E_1)$ we denote the totality of all functions $F(t)$ from $C_\infty(E_1)$ that are finite near zero.

Lemma 2. In the space $S_p(E)$ ($1 \leq p < \infty$) the set $C_\infty(E_1)$ is dense. In the subspace $S_p^0(E)$ the set $C_\infty^0(E_1)$ is dense.

3. Let A be a strongly positive operator in E . This means (see (1)) that the operator $-A$ is the generator of an analytic semigroup e^{-tA} , whose norm decreases exponentially.

We shall say that $v \in E$ belongs to the set E_α ($0 \leq \alpha \leq 1$) if

$$|v|_\alpha = \|e^{-tA}v\|_S \frac{1}{1-\alpha} < \infty.$$

The set E_α is a Banach space with norm (3) $|v|_\alpha$. The spaces E_α are closely connected with the domains of definition $D(A^\alpha)$ of fractional powers A^α (for the definition of A^α , see (5)).

Lemma 3. If $v \in E_\alpha$ ($\alpha > 0$), then $v \in D(A^{\alpha-\varepsilon})$ ($\varepsilon > 0$), and

$$\|A^{\alpha-\varepsilon}v\|_E \leq K\varepsilon^{-1}|v|_\alpha.$$

If $v \in D(A^{\alpha+\varepsilon})$ ($0 \leq \alpha < 1$, $\varepsilon > 0$), then $v \in E_\alpha$ and

$$|v|_\alpha \leq K\varepsilon^{-1}\|A^{\alpha+\varepsilon}v\|_E.$$

4. **Theorem 1.** Let $f(t) \in S_p(E)$ ($1 < p < \infty$), $v_0 \in D(A)$, and $Av_0 - f(0) \in E_{1/q}$. Then there exists a unique absolutely continuous solution $v(t)$ of problem (1) such that the functions $v'(t)$ and $Av(t)$ belong to $S_p(E)$, $v'(t)$ is a continuous function with values in $E_{1/q}$, and the inequality

$$\|v'(t)\|_{S_p} + \|Av(t)\|_{S_p} + \max_{0 \leq t \leq 1} |v'(t)|_{1/q} \leq Cpq(\|f(t)\|_{S_p} + |f(0) - Av(0)|_{1/q}) \quad (4)$$

holds, where the constant C depends neither on f nor on v_0 .

If, moreover, $f(t)$ is a continuous function with values in $E_{1/q}$, then $Av(t)$ is a continuous function with values in $E_{1/q}$, and the inequality

$$\|v'(t)\|_{S_p} + \|Av(t)\|_{S_p} + \max_{0 \leq t \leq 1} |v'(t)|_{1/q} + \max_{0 \leq t \leq 1} |Av(t)|_{1/q} \leq \tilde{C}pq \left(\|f(t)\|_{S_p} + \max_{0 \leq t \leq 1} |f(t)|_{1/q} + |Av_0|_{1/q} \right) \quad (5)$$

holds with a constant $\tilde{C}$ depending neither on f nor on v_0 .

5. To each function $f(t) \in C_0(E)$ we assign the number

$$\|f(t)\|_{S'_p} = \|F(t)\|_{S_p}, \quad F(t) = \int_0^t f(s) ds. \quad (6)$$

The completion of $C_0(E)$ with respect to the norm $\|f(t)\|_{S'_p}$ is a Banach space $S'_p(E)$. Roughly speaking, this space consists of generalized derivatives, in the sense of S. L. Sobolev–L. Schwartz, of functions from the space $S_p(E)$. It follows from Lemma 2 that in the space $S'_p(E)$, for $p < \infty$, the set $C_\infty(E_1)$ is dense.

The operator d/dt , acting from $S_p(E)$ to $S'_p(E)$ and defined in the natural way on the everywhere dense set $C_\infty^0(E_1)$ in $S_p^0(E)$, can be extended by continuity to an operator defined on all of $S_p^0(E)$ and mapping $S_p^0(E)$ one-to-one onto $S'_p(E)$. We shall also denote the extended operator by d/dt . This operator can also be extended to all of $S_p(E)$, if one sets

$$\frac{d}{dt}F(t) \equiv \frac{d}{dt}[F(t) - F(0)]$$

for any function $F(t) \in S_p(E)$. Now d/dt no longer maps $S_p(E)$ one-to-one onto $S'_p(E)$, but from the equality

$$\frac{d}{dt}F(t) \equiv 0$$

it follows that

$$F(t) \equiv F(0).$$

The domain of definition of the operator d/dt in $S_p(E)$ will be denoted by $D_p(d/dt)$.

Let A be an operator acting from $S_p(E)$ to $S'_p(E)$, defined

naturally on the set $C_\infty(D_E[A])$, where $D_E[A]$ is the domain of definition of the operator A as an operator in E . The operator A thus defined from $S_p(E)$ into $S'_p(E)$ admits a closure, for which we shall keep the same notation A , and by $D_p(A)$ we shall denote its domain of definition.

Finally, let $D_p = D_p(d/dt) \cap D_p(A)$.

Lemma 4. Let the function $v(t)$ belong to D_p ($1 < p < \infty$). Then $v(t)$ is a continuous function with values in $E_{1/q}$.

6. **Theorem 2.** Let $f(t) \in S'_p(E)$ ($1 < p < \infty$), $v_0 \in E_{1/q}$. Then there exists a unique solution $v(t)$ of problem (1) from D_p , and the inequality

$$\|v'(t)\|_{S'_p} + \|Av(t)\|_{S'_p} + \max_{0 \leq t \leq 1} \|v(t)\|_{1/q} \leq Cpq(\|f(t)\|_{S'_p} + \|v_0\|_{1/q}); \quad (7)$$

holds, where the constant C depends neither on f nor on v_0 .

7. A function $F(t)$ belonging to the Bochner space $B_r([0, 1], E)$ is an element of the Banach space $S_{p,r}(E)$ ($1 \leq p \leq \infty$, $1 \leq r \leq \infty$) if

$$\|F(t)\|_{S_{p,r}} \equiv \left(\int_0^1 \left\{ \int_0^1 [\psi(t, \tau) \|F(t)\|_E]^r dt \right\}^{p/r} d\tau \right)^{1/p} + \left(\int_0^1 \left\{ \int_0^{1-\tau} \|F(t+\tau) - F(t)\|_E^r dt \right\}^{p/r} \tau^{-p} d\tau \right)^{1/p} < \infty$$

where $\psi(t, \tau) = \min(1/t, 1/\tau)$. In $S_{p,r}(E)$ for ($1 \leq p < \infty$, $1 \leq r < \infty$) the set $C_\infty(E_1)$ is dense. The closure of the set $C_\infty^0(E_1)$ in the norm of the space $S_{p,r}(E)$ forms the subspace $S_{p,r}^0(E)$.

Lemma 5. If $1/p + 1/q > 1$, then the subspace $S_{p,r}^0(E)$ coincides with the whole space $S_{p,r}(E)$.

8. The closure of the set $D(A)$ in the norm

$$|v|_{p,r} \equiv \|e^{-tA}v\|_{S_{p,r}} \quad (9)$$

forms the Banach space $E_{p,r}$. The closure of $D(A^2)$ in the norm

$$|v|_{p,r,1} \equiv \|Ae^{-tA}v\|_{S_{p,r}} \quad (10)$$

forms the Banach space $E_{p,r,1}$. The spaces $E_{p,r}$ and $E_{p,r,1}$, as well as the spaces E_α (see item 3), are closely connected with the spaces $D(A^\alpha)$. Below, by the space $D(A^\alpha)$ for $\alpha < 0$ is meant the completion $\bar{E}$ in the norm $\|A^\alpha v\|_E$.

Lemma 6. Let $\alpha > 1/p + 1/r - 1 = s$. If $v \in D(A^\alpha)$, then $v \in E_{p,r}$ and

$$|v|_{p,r} \leq C(\alpha - s)^{-1} \|A^\alpha v\|_E.$$

If $v \in D(A^{\alpha+1})$, then $v \in E_{p,r,1}$ and

$$|v|_{p,r,1} \leq C(\alpha - s)^{-1} \|A^{\alpha+1}v\|_E.$$

Let $\alpha < s$. If $v \in E_{p,r}$, then $v \in D(A^\alpha)$ and

$$\|A^\alpha v\|_E \leq C(s - \alpha)^{-1} |v|_{p,r}.$$

If $v \in E_{p,r,1}$, then $v \in D(A^{\alpha+1})$ and

$$\|A^{\alpha+1}v\|_E \leq C(s - \alpha)^{-1} |v|_{p,r,1}.$$

9. **Theorem 3.** Let $f(t) \in S_{p,r}^0(E)$ ($1 < p < \infty$), $v_0 \in E_{p,r,1}$. Then there exists a unique absolutely continuous solution $v(t)$ of problem (1) such that the functions $v'(t)$ and $Av(t)$ belong to $S_{p,r}(E)$, and the inequality

$$\|v'(t)\|_{S_{p,r}} + \|Av(t)\|_{S_{p,r}} \leq Cpq(\|f(t)\|_{S_{p,r}} + |v_0|_{p,r,1}), \quad (11)$$

holds, where the constant C depends neither on f nor on v_0 .

10. Similarly to how the space of generalized functions $S'_p(E)$ was constructed, the space $S_{p,r}^0(E)$ can be constructed and the operators d/dt and A , acting in this space, can be defined with domains of definition $D_{p,r}(d/dt)$ and $D_{p,r}(A)$. Put

$$D_{p,r} = D_{p,r}(d/dt) \cap D_{p,r}(A).$$

Theorem 4. Let $f(t) \in S_{p,r}(E)$ ($1 < p < \infty$), $v_0 \in E_{p,r}$. Then there exists a unique solution $v(t)$ of problem (1) from $D_{p,r}$, and the inequality

$$\|v'(t)\|_{S_{p,r}} + \|Av(t)\|_{S_{p,r}} \leq Cpq(\|f(t)\|_{S_{p,r}} + |v_0|_{p,r}), \quad (12)$$

holds, where the constant C depends neither on f nor on v_0 .

11. **Remark 1.** For the spaces E_α , $E_{p,r}$, and $E_{p,r,1}$ there are interpolation theorems analogous to those given in (2).

Remark 2. In defining the spaces S_p and $S_{p,r}$, instead of the L_p -norm one could have taken the norm of a Lorentz space (cf. (4)). In defining the spaces $S_{p,r}$, instead of the L_r -norm one could have taken a more general norm, for example the Orlicz norm.

Remark 3. The results obtained for equations with a constant operator carry over without great difficulty to equations with a variable operator having a constant domain of definition.

A detailed consideration of the questions touched upon in these remarks will be devoted to another paper by the author.

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REFERENCES

1. P. E. Sobolevskii, DAN, **157**, No. 1 (1964).
2. P. E. Sobolevskii, UMN, **19**, issue 6 (120) (1964).
3. P. E. Sobolevskii, DAN, **165**, No. 3 (1965).

4. P. E. Sobolevskii, UMN, **21**, issue 1 (127) (1966).

5. M. A. Krasnosel' skii, P. E. Sobolevskii, DAN, **129**, No. 3 (1959).

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