

SOME THEOREMS CONCERNING THE DIMENSION Ind

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Abstract

Full Text

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MATHEMATICS

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SOME THEOREMS CONCERNING THE DIMENSION Ind

(Presented by Academician P. S. Aleksandrov on 13 I 1967)

Dauker ⁽¹⁾ proves sum theorems for Ind^* for hereditarily normal spaces. In this note we free ourselves from the condition of hereditary normality ^{**} of the spaces participating in the sum, but impose conditions on the intersections of these spaces.

Theorem 1. *If a normal space $P = X \cup Y$ is the sum of two closed subsets X and Y such that: 1) $\text{Ind } X \leq n$, $\text{Ind } Y \leq n$; 2) $\text{Ind}(X \cap Y) \leq 0$, then $\text{Ind } P \leq n$.*

Proof. Let $F = X \cap Y$. Take a closed set Φ in P and an arbitrary neighborhood $O\Phi$ of it.

- a) If $\Phi \cap F = \emptyset$, then for $O\Phi$ there will be found a neighborhood $O_1\Phi$ of the set Φ such that $[O_1\Phi] \cap F = \emptyset$, $O_1\Phi \subseteq O\Phi$, and $\text{Ind fr } O_1\Phi \leq n - 1$.
- b) Let now $\Phi \cap F \neq \emptyset$. Put $\Phi_1 = \Phi \cap F$, $O_F\Phi_1 = O\Phi \cap F$, where the subscript F indicates in which set this $O_F\Phi_1$ is open.

Since $\text{Ind } F \leq 0$, there will be found an open-and-closed neighborhood $O_F^1\Phi_1$ of the set Φ_1 , contained in $O_F\Phi_1$.

Consider the sets $\Phi_2 = \Phi \cup O_F^1\Phi_1$ and $\Phi'_2 = (P \setminus O\Phi) \cup F_1$, where $F_1 = F \setminus O_F^1\Phi_1$. The sets Φ_2 and Φ'_2 are closed in P , and $\Phi_2 \cap \Phi'_2 = \emptyset$. Next take the sets $X_1 = X \cap \Phi_2$, $X'_1 = X \cap \Phi'_2$ and the sets $Y_1 = Y \cap \Phi_2$, $Y'_1 = Y \cap \Phi'_2$. The sets X_1, X'_1 (Y_1, Y'_1) are closed in X (Y) and $X'_1 \cap X_1 = \emptyset$ ($Y'_1 \cap Y_1 = \emptyset$); therefore, in view of the fact that $\text{Ind } X \leq n$, $\text{Ind } Y \leq n$, there will be found a neighborhood Γ_1 (Γ_2) of the set X_1 (Y_1) in X (Y) such that $\text{Ind fr } \Gamma_1 \leq n - 1$ ($\text{Ind fr } \Gamma_2 \leq n - 1$) and $[\Gamma_1]_X \cap X'_1 = \emptyset$ ($[\Gamma_2]_Y \cap Y'_1 = \emptyset$). From the choice of the sets Γ_1 and Γ_2 it is clear that the set $\Gamma = \Gamma_1 \cup \Gamma_2$ is open in P , since $\Gamma_1 \cap Y = \Gamma_2 \cap X$. Moreover, $\text{fr}_P \Gamma_1 \cap \text{fr}_P \Gamma_2 = \emptyset$, since $\text{fr}_P \Gamma_1 \subset X$, $\text{fr}_P \Gamma_2 \subset Y$ and $\text{fr}_P \Gamma_1 \cap F = \text{fr}_P \Gamma_2 \cap F = \emptyset$, while $\text{fr}_P \Gamma_1 \cup \text{fr}_P \Gamma_2$ is closed. Then $\text{Ind fr}_P \Gamma \leq n - 1$ and $\Phi \subset \Gamma \subset O\Phi$. Thus theorem 1 is proved.

From theorem 1 it follows

Theorem 2. *Let a normal space*

$$P = \bigcup_{i=1}^n X_i,$$

where: 1) X_i is closed in P ; 2) $\text{Ind}(X_i \cap X_j) \leq 0$ for $i \neq j$ ($i = 1, 2, \dots, n$; $j = 1, 2, \dots, n$); 3) $\text{Ind } X_i \leq n$ ($i = 1, 2, \dots, n$). Then $\text{Ind } P \leq n$.

Using theorem 2, we shall now prove theorem 3.

Theorem 3. *Let $P = X \times Y$, where X, Y are bicomacts and $\text{Ind } X \leq 1$, $\text{Ind } Y \leq 1$. Then $\text{Ind } P \leq 2$.*

We shall first prove a lemma.

Lemma 1. *Let X be a one-dimensional bicomact, $\text{Ind } X \leq 1$, and let $\omega = \{O_1, O_2, \dots, O_s\}$ be an arbitrary open covering. Then one can construct a refinement $\alpha = \{A_1, A_2, \dots, A_s\}$, inscribed in ω , satisfying the condition $\text{Ind fr}(\text{Int } A_j) \leq 0$.**

* A space X has $\text{Ind } X \leq n$ if for every closed set F and every neighborhood OF of it there exists $O_1 F \subset OF$ such that $\text{Ind fr } O_1 F \leq n - 1$, $\text{Ind } \emptyset = -1$.

** All spaces considered in the note are assumed to be Hausdorff and normal.

*** A covering $\alpha = \{A_1, A_2, \dots, A_s\}$ of a space X is called a *partition* if $A_i = [\text{Int } A_i]$ and $\text{Int } A_i \cap \text{Int } A_j = \emptyset$ for $i \neq j$ ($i, j = 1, 2, \dots, n$).

Proof. Take an arbitrary open cover $\omega = \{O_1, O_2, \dots, O_s\}$. In ω inscribe such an open cover $\omega^1 = \{O_1^1, O_2^1, \dots, O_s^1\}$ that $[O_i^1] \subseteq O_i$ and $\text{Ind Fr } O_i^1 \leq 0$, $i = 1, 2, \dots, s$.

Define by induction the sets A_i ($i = 1, 2, \dots, s$),

$$\text{Int } A_1 = O_1^1 \setminus \bigcup_{i=2}^s [O_i^1], \quad A_1 = [\text{Int } A_1],$$

then

$$A_1 \cup \bigcup_{i=2}^s O_i^1 = X \quad \text{and} \quad \text{Ind Fr}(\text{Int } A_1) \leq 0,$$

since

$$\text{Fr}(\text{Int } A_1) \subseteq \bigcup_{i=1}^s \text{Fr } O_i^1.$$

Suppose that we have already constructed sets $A_1, A_2, \dots, A_{k-1}$ satisfying the conditions:

$$\text{a) } \quad \text{Int } A_{i_1} \cap \text{Int } A_{i_2} = \emptyset \quad \text{for } i_1 \neq i_2 \quad (i_1, i_2 = 1, 2, \dots, k-1);$$

$$\begin{aligned} \text{b)} \quad & \bigcup_{j=1}^{k-1} A_j \cup \bigcup_{i=k}^s O_i^1 = X; \\ \text{c)} \quad & \text{Fr Int } A_j \subseteq \bigcup_{i=1}^s \text{Fr } O_i^1 \quad (j = 1, 2, \dots, k-1). \end{aligned}$$

Construct A_k :

$$\text{Int } A_k = O_k^1 \setminus \left(\bigcup_{i=k+1}^s [O_i^1] \cup \bigcup_{j=1}^{k-1} A_j \right), \quad A_k = [\text{Int } A_k].$$

Then, by the definition of the set A_k , we obtain:

$$\text{a)} \quad \bigcup_{j=1}^k A_j \cup \bigcup_{i=k+1}^s O_i^1 = X.$$

Further,

$$\text{Fr Int } A_k \subseteq \bigcup_{i=k}^s \text{Fr } O_i^1 \cup \bigcup_{i=1}^{k-1} \text{Fr Int } A_j,$$

but

$$\text{Fr Int } A_j \subseteq \bigcup_{i=1}^s \text{Fr } O_i^1 \quad (j = 1, 2, \dots, k-1),$$

therefore

$$\text{b)} \quad \text{Fr Int } A_k \subseteq \bigcup_{i=1}^s \text{Fr } O_i^1$$

and

$$\text{c)} \quad \text{Int } A_k \cap \text{Int } A_j = \emptyset \quad (j = 1, 2, \dots, k-1).$$

The partition $\alpha = \{A_1, A_2, \dots, A_s\}$ is the desired one. Indeed, α is inscribed in ω , and

$$\text{Ind Fr}(\text{Int } A_j) \leq \text{Ind} \left(\bigcup_{i=1}^s \text{Fr } O_i^1 \right) \leq 0,$$

since $\text{Ind Fr } O_i^1 \leq 0$.

Proof of Theorem 3. Let $P = X \times Y$, where X, Y are bicomacta and $\text{Ind } X \leq 1, \text{Ind } Y \leq 1$. Let a closed set $F \subset P$ and its neighborhood OF be arbitrary. One may take a cover $\Omega = \{\omega_1 \times \omega_2\}$, whose elements are the sets $\{O_i \times G_j\}$ ($i = 1, 2, \dots, n; j = 1, 2, \dots, m$), where $\omega_1 = \{O_1, \dots, O_n\}$ is an open cover of the bicomactum X , and $\omega_2 = \{G_1, \dots, G_m\}$ is an open cover of the bicomactum Y . In this case Ω can be chosen so that the following conditions will be fulfilled:

$$1) \quad \text{if } (O_i \times G_j) \cap F = \emptyset, \text{ then } [(O_i \times G_j)] \cap F = \emptyset;$$

$$2) \quad \left[\bigcup_{i=1}^n \bigcup_{j=1}^m (O_j \times G_j) \right] \subseteq OF, \quad \text{where } [O_i \times G_j] \cap F \neq \emptyset.$$

Now, using Lemma 1, construct such partitions

$$\alpha = \{A_1, A_2, \dots, A_n\} \quad \text{and} \quad \beta_1 = \{B_1, B_2, \dots, B_m\}$$

of the spaces X and Y , respectively, that: a) α_1 is inscribed in ω_1 , β is inscribed in ω_2 ; b) $\text{Ind Fr}(\text{Int } A_i) \leq 0$ and $\text{Ind Fr}(\text{Int } B_j) \leq 0$ ($i = 1, 2, \dots, n$; $j = 1, 2, \dots, m$).

The partition

$$\alpha = \{\alpha_1 \times \alpha_2\},$$

consisting of the sets

$$A_i \times B_j \quad (i = 1, 2, \dots, n; j = 1, 2, \dots, m)$$

of the bicom pactum P , has the property that

$$G = P \setminus \bigcup A_i \times B_j$$

(where $(A_i \times B_j) \cap F = \emptyset$) is open in P and $G \subseteq OF$. The boundary of G consists of closed subsets of sets of the form

$$\text{Fr } A_i \times B_j \quad \text{and} \quad A_i \times \text{Fr } B_j \quad (i = 1, 2, \dots, n; j = 1, 2, \dots, m),$$

and moreover these sets are such that if two of them intersect, then they intersect in a zero-dimensional closed set. Therefore, from Theorem 2 it follows that

$$\text{Ind Fr } G \leq 1.$$

Theorem 3 is proved.

Remark 1. Note that Lemma 1 is true for any normal space X , provided only that $\text{Ind } X \leq 1$.

We now introduce two definitions.

Definition 1. We shall say that a closed set F is **normally situated** in a normal space X , if for every closed set $\Phi \subseteq F$ the subspace $X \setminus \Phi$ is normal.

Example. Let S be the bicom pactum of O. V. Lokutsievskii ⁽²⁾. In this bicom pactum the set $I \times \omega_1$ has the property that $S \setminus (I \times \omega_1)$ is normal, but the set $I \times \omega_1$ does not lie normally in the bicom pactum S . If, however, a closed set $F \subseteq S$ is such that $F \cap (I \times \omega_1) = \emptyset$, then F lies normally in S .

Definition 2. A closed set F will be called **normally Ind-lying** in a normal space X if, for every closed set $\Phi \subseteq F$, the subspace $X \setminus F$ is normal and

$$\text{Ind}(X \setminus \Phi) \leq \text{Ind } X.$$

Theorem 4. *If a normal space P is the sum of two closed subsets X and Y in it such that: 1) $\text{Ind } X \leq n$, $\text{Ind } Y \leq n$; 2) $F = X \cap Y$ is a set normally Ind-lying in X and in Y ; 3) $\text{Ind } F \leq n$, then $\text{Ind } P \leq n$.*

Before proving Theorem 4, we prove Theorem 5, from which Theorem 4 will easily follow.

Theorem 5. *Let P be a normal space. If in P there exists a closed set F such that $\text{Ind } F \leq n$, $\text{Ind}(P \setminus F) \leq n$, and F lies normally in P , then $\text{Ind } P \leq n$.*

Proof. We shall prove the theorem by induction on the number n . For the sum of empty sets this is obvious. Suppose that Theorem 5 has already been proved for $n - 1$, i.e. if P is a normal space and in P there exists a closed set F , normally lying in P , such that $\text{Ind } F \leq n - 1$ and $\text{Ind}(P \setminus F) \leq n - 1$, then $\text{Ind } P \leq n - 1$. We now prove Theorem 5 for n .

Let Φ be closed in P , and let $O\Phi$ be an arbitrary neighborhood of it.

a) If $\Phi \cap F = \emptyset$, then there exists a neighborhood $O_1\Phi \subseteq O\Phi$ such that

$$\text{Ind Fr } O_1\Phi \leq n - 1.$$

b) Let $\Phi \cap F \neq \emptyset$. Denote by Φ_1 and $O\Phi_1$ the sets

$$\Phi_1 = \Phi \cap F, \quad O\Phi_1 = O\Phi \cap F.$$

Since $\text{Ind } F \leq n$, choose a neighborhood $O_1\Phi_1 \subseteq O\Phi_1$ of the set Φ_1 such that

$$\text{Ind Fr}_F O_1\Phi_1 \leq n - 1.$$

Put

$$\Gamma = F \setminus [O_1\Phi_1], \quad G_1 = P \setminus \text{Fr}_F O_1\Phi_1.$$

Then G_1 is a normal space, Γ and $O_1\Phi_1$ are closed in G_1 , and $\Gamma \cap O_1\Phi_1 = \emptyset$, $\text{Ind}(P \setminus F) \leq n$. Consider the sets

$$\Phi_2 = \Phi \cup O_1\Phi_1, \quad \Phi'_2 = \Gamma \cup (G_1 \setminus O\Phi).$$

We shall have $\Phi_2 \cap \Phi'_2 = \emptyset$, and Φ_2, Φ'_2 are closed in G_1 . Now choose open sets G_2 and G'_2 in G_1 such that

$$\Phi_2 \subseteq G_2, \quad \Phi'_2 \subseteq G'_2, \quad [G_2]_{G_1} \cap [G'_2]_{G_1} = \emptyset.$$

Next consider the sets

$$G_3 = G_2 \setminus F, \quad G'_3 = G'_2 \setminus F.$$

We have

$$[G_3]_{(P \setminus F)} \cap [G'_3]_{(P \setminus F)} = \emptyset.$$

Since $\text{Ind}(P \setminus F) \leq n$, choose an open set G_4 in $P \setminus F$ such that

$$[G_3]_{(P \setminus F)} \subseteq G_4, \quad [G_4]_{(P \setminus F)} \cap [G'_3]_{(P \setminus F)} = \emptyset$$

and

$$\text{Ind Fr}_{(P \setminus F)} G_4 \leq n - 1.$$

The set

$$G_5 = G_4 \cup O_1 \Phi_1$$

is open in P and

$$\Phi \subseteq G_5 \subseteq O\Phi.$$

Notice that

$$\text{Fr}_P G_5 = \text{Fr}_{(P \setminus F)} G_4 \cup \text{Fr}_F O_1 \Phi_1,$$

where

$$\text{Ind Fr}_F O_1 \Phi_1 \leq n - 1,$$

$\text{Fr}_F O_1 \Phi_1$ lies normally in $\text{Fr}_P G_5$, and

$$\text{Ind}(\text{Fr}_P G_5 \setminus \text{Fr}_F O_1 \Phi_1) = \text{Ind Fr}_{(P \setminus F)} G_4 \leq n - 1.$$

Thus, $\text{Ind Fr}_P G_5 \leq n - 1$. Theorem 5 is proved.

Remark 2. The bicomactum of Lokutsievskii ⁽²⁾ S is represented as the sum of the closed set $F = I \times \omega_1$ and the normal set $(S \setminus F)$, $\text{Ind } F \leq 1$, $\text{Ind}(S \setminus F) \leq 1$, while $\text{Ind } S = 2$. But, as was said in Remark 1, the set F does not lie normally in S . Thus, the concept of a normally lying set is essential.

We now prove Theorem 4. Let

$$P = X \cup Y, \quad F = X \cap Y.$$

Let a closed set $\Phi \subseteq P$ and its neighborhood $O\Phi$ be arbitrary.

a) If $\Phi \cap F = \emptyset$, then there exists a neighborhood $O_1 \Phi \subseteq O\Phi$ such that

$$\text{Ind Fr } O_1 \Phi \leq n - 1.$$

b) Let

$$\Phi \cap F = \Phi_1 \neq \emptyset.$$

Put

$$O\Phi_1 = F \cap O\Phi.$$

Choose $O_1 \Phi_1 \subseteq O\Phi_1$ in such a way that

$$\text{Ind Fr}_F O_1 \Phi_1 \leq n - 1.$$

Let

$$F_1 = F \setminus [O_1 \Phi_1].$$

Then F_1 and $O_1 \Phi_1$ are closed in

$$G = P \setminus \text{Fr}_F O_1 \Phi_1$$

and

$$F_1 \cap O_1 \Phi_1 = \emptyset.$$

Consider now the sets

$$\Phi_2 = \Phi \cup O_1 \Phi_1, \quad \Phi'_2 = F_2 \cup (G \setminus O\Phi).$$

The sets Φ_2, Φ'_2 are closed in G and

$$\Phi_2 \cap \Phi'_2 = \emptyset.$$

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now take the sets $X_1 = X \cap \Phi_2$, $X'_1 = X \cap \Phi'_2$ and $Y_1 = Y \cap \Phi_2$, $Y'_1 = Y \cap \Phi'_2$, $G_X = X \cap G$, $G_Y = Y \cap G$. We have $G_X = X \setminus \text{bd}_F O_1 \Phi_1$, $G_Y = Y \setminus \text{bd}_F O_1 \Phi_1$; consequently, G_X, G_Y are normal and $\text{Ind } G_X \leq n$, $\text{Ind } G_Y \leq n$. Therefore we can take an open subset $\Gamma_X (\Gamma_Y)$ in $G_X (G_Y)$ such that $X_1 \subset \Gamma_X (Y_1 \subset \Gamma_Y)$, $[\Gamma_X] \cap X'_1 = \emptyset$, $([\Gamma_Y] \cap Y'_1 = \emptyset)$, and $\text{Ind } \text{bd}_{G_X} \Gamma_X \leq n-1$ ($\text{Ind } \text{bd}_{G_Y} \Gamma_Y \leq n-1$). The set $\Gamma = \Gamma_X \cup \Gamma_Y$ is open in G , and hence open also in P , and

$$\text{bd}_P \Gamma = \text{bd}_{G_X} \Gamma_X \cup \text{bd}_{G_Y} \Gamma_Y \cup \text{bd}_F O_1 \Phi_1,$$

where

$$\text{bd}_{G_X} \Gamma_X \cup \text{bd}_{G_Y} \Gamma_Y = \emptyset,$$

therefore

$$\text{Ind}(\text{bd}_{G_X} \Gamma_X \cap \text{bd}_{G_Y} \Gamma_Y) \leq n-1$$

and

$$\text{Ind } \text{bd}_F O_1 \Phi_1 \leq n-1.$$

Further, we see that $\text{bd}_F O_1 \Phi_1$ lies normally in $\text{bd}_P \Gamma$ and

$$\text{bd}_F O_1 \Phi_1 \cap (\text{bd}_{G_X} \Gamma_X \cup \text{bd}_{G_Y} \Gamma_Y) = \emptyset;$$

consequently, by Theorem 5,

$$\text{Ind } \text{bd}_P \Gamma \leq n-1.$$

Theorem 4 is proved. From Theorems 4 and 5 it follows that

Theorem 5' (Dowker ⁽¹⁾). *Let P be a hereditarily normal space. If $F \subset P$ is closed and $\text{Ind } F \leq n$, $\text{Ind}(P \setminus F) \leq n$, then $\text{Ind } P \leq n$.*

Theorem 4'. *Let $P = X \cup Y$ be a hereditarily normal space; X, Y are closed in P and $\text{Ind}(X \cap Y) \leq n$, $\text{Ind}(X \setminus F) \leq n$, $\text{Ind}(Y \setminus F) \leq n$, where $F = X \cap Y$. Then $\text{Ind } P \leq n$.*

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CITED LITERATURE

¹ C. H. Dowker, Quart. J. Math. Oxford, **4**, 267 (1953).

² O. V. Lokutsievskii, DAN, **67**, No. 2 (1949).

Note: Figure translations are in progress. See original paper for figures.

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