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A. P. Savin

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Abstract

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MATHEMATICS

A. P. SAVIN

ON THE STRUCTURE OF THE SET OF POINTS OF MUTUAL UNIQUENESS UNDER CONTINUOUS MAPPINGS OF MANIFOLDS

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Let f be a continuous mapping of an n -dimensional closed manifold M^n onto an n -dimensional closed manifold μ^n with degree c . We shall assume that the degree of the mapping is, in absolute value, greater than 1. A point $\xi \in \mu^n$ will be called a **point of mutual uniqueness** of the mapping f if its full inverse image consists of a single point. The set of all points of mutual uniqueness of the mapping f will be denoted by Ξ , and its full inverse image $f^{-1}(\Xi)$ by X .

In the present article the structure of this set is considered. The set of points of mutual uniqueness under mappings of n -dimensional manifolds was studied in the work of G. Hopf ⁽¹⁾, in which the case where the mapping f is simplicial was examined in detail. It was shown that $\dim \Xi \leq n - 2$, and the $(n - 2)$ -nd Betti number of the set Ξ was estimated in terms of the Betti numbers of the indicated manifolds. For $n = 2$ it was proved that, under continuous mappings, the set Ξ consists of a finite number of points, and this number was estimated in terms of the Euler characteristics of the manifolds M^n and μ^n .

Below the following propositions will be proved:

Theorem 1. The sets X and Ξ are homeomorphic.

Theorem 2. The set Ξ has type G_δ .

Theorem 3. The set Ξ does not separate any ball in μ^n .

Theorem 4. For $n = 3$, the set Ξ contains no locally connected continuum of index greater than two.

It follows from Theorem 1 that the assertions corresponding to Theorems 2–4 also hold for the set X .

Proof of Theorem 1. By the definition of the sets X and Ξ , the mapping $f : X \rightarrow \Xi$ is one-to-one and continuous. To establish the homeomorphy of these sets it is enough to establish the continuity of the mapping $f^{-1} : \Xi \rightarrow X$.

Let $\{\xi_n\}$ be a sequence of points of Ξ converging to the point $\xi_0 \in \Xi$. Consider the sequence $\{f^{-1}(\xi_n)\}$ in the set X , and let x be one of the cluster points

of this sequence (such a point always exists, since M^n is compact). From the continuity of the mapping f it follows that $f(x) = \xi_0$, and since ξ_0 is a point of mutual uniqueness, $f^{-1}(\xi_0) = x$; consequently, x is the unique cluster point of the sequence $\{f^{-1}(\xi_n)\}$, and the mapping $f^{-1} : \Xi \rightarrow X$ is continuous at the point ξ_0 , and, by the arbitrariness of the choice of ξ_0 , everywhere on the set Ξ .

Proof of Theorem 2. First note that for any neighborhood $O(x)$ of a point $x \in M^n$ there is a neighborhood $O(\xi)$ of the point $\xi = f(x)$ possessing the property that $O(\xi)$ does not intersect the set $f(M^n \setminus O(x))$. Indeed, otherwise there would exist a sequence $\{x_n\}$ such that $x_n \in M^n \setminus O(x)$ and $\rho(f(x_n), \xi) \rightarrow 0$ as $n \rightarrow \infty$. Then, from the continuity of the mapping f , it follows that for any limit point x_0 of the sequence $\{x_n\}$ the relation $f(x_0) = \xi$ holds, which is impossible, since $f^{-1}(\xi)$ cannot coincide with the point x_0 .

We shall now construct, for each point $x \in X$, a sequence of its neighborhoods $\{O(x, 1/n)\}$, where $O(x, 1/n) = \{y \mid \rho(x, y) < 1/n\}$. Let $O_n(\xi)$ be a neighborhood of the point $\xi = f(x)$, chosen as stated for the neighborhood $O(x, 1/n)$. Then

$$A_n = \bigcup_{\xi \in \Xi} O_n(\xi)$$

is an open set containing the set Ξ . Obviously,

$$\bigcap_{n=1}^{\infty} A_n \supset \Xi.$$

We shall show that

$$\bigcap_{n=1}^{\infty} A_n \subset \Xi.$$

Let $\eta \in \mu^n \setminus \Xi$; then the diameter of the full preimage of this point is nonzero; denote it by $d(\eta)$. None of the sets $O(x, 1/n)$, for $2/n < d(\eta)$, can contain the full preimage of the point η ; consequently, by construction, the point η is not contained in any of the A_n for $n > 2/d(\eta)$, which proves our assertion. Thus,

$$\Xi = \bigcap_{n=1}^{\infty} A_n,$$

where all A_n are open in μ^n , and, consequently, it is of type G_δ .

In the proof of Theorems 1 and 2 we nowhere used the circumstance that μ^n and M^n are manifolds, and we imposed no restrictions on the degree of the mapping f . Therefore the indicated theorems are also valid for discontinuous mappings of metric spaces.

Proof of Theorem 3. We shall argue by contradiction. Suppose the set Ξ separates some ball T^n in the manifold μ^n . Then, as is known, there exists a closed set $\Psi \subset \Xi$ that also separates this ball. Denote by A_1 and A_2 the open subsets of T^n into which the set Ψ separates T^n . In the set $K = \Psi \cap T^n$, consider

the subsets $K_1 = K \cap [A_1]$ and $K_2 = K \cap [A_2]$. Observe that $K_1 \cup K_2 = K$, for otherwise there would be a point $\xi \in K$ lying at a nonzero distance from both the set A_1 and the set A_2 ; in that case the set K , and hence also Ξ , would contain an n -dimensional ball, and the degree of the mapping f , contrary to the assumption, would be equal to $+1$ or -1 .

Next, the set $K_1 \cap K_2$ is nonempty. Otherwise the sets $K_1 \cup A_1$ and $K_2 \cup A_2$ would be open, would not intersect, and together would make up the connected set T^n .

Now choose in the set $K_1 \cap K_2$ an arbitrary point η , and consider in the manifold M^n a ball neighborhood $O(x)$ of the point $x = f^{-1}(\eta)$ such that $f(O(x)) \subset T^n$. The set $f^{-1}(K) = W$ separates $O(x)$ into two open sets:

$$U = O(x) \cap f^{-1}(A_1)$$

and

$$V = O(x) \cap f^{-1}(A_2),$$

and, according to the choice of the point η , both sets U and V are nonempty.

Consider a discontinuous mapping $\varphi_0 : M^n \rightarrow S^n$, where S^n is the n -dimensional sphere, which contracts the set $M^n \setminus O(x)$ to a point $a \in S^n$ and maps $O(x)$ onto $S^n \setminus a$ homeomorphically. The degree of this mapping is obviously equal to $+1$ or -1 .

We shall regard the sphere S^n as the unit sphere with center at the origin of the $(n+1)$ -dimensional Euclidean coordinate space E^{n+1} ; denote by S^{n-1} its intersection with the hyperplane $x_1 = 0$, and by E^+ and E^- the intersections of S^n with the half-spaces $x_1 > 0$ and $x_1 < 0$. We denote the points

$$(0, 0, \dots, 0, +1)$$

and

$$(0, 0, \dots, 0, -1)$$

by a^+ and a^- , respectively.

From results of the author [2] it follows that there exists a discontinuous deformation φ_t of the mapping φ_0 such that

$$\varphi_1((M^n \setminus O(x)) \cup W) = S^{n-1},$$

and, moreover,

$$\varphi_1(x) = a^+, \quad \varphi_1(M^n \setminus O(x)) = a^-,$$

besides,

$$\varphi_1(U) = E^+, \quad \varphi_1(V) = E^-.$$

Since the mappings φ_0 and φ_1 are homotopic, their degrees are equal.

The mapping

$$F = \varphi_1 f^{-1} : \Psi \rightarrow S^{n-1}$$

is discontinuous, since the mapping f^{-1} is discontinuous by Theorem 1, and the mapping φ_1 is so by construction. Extend the mapping F to the closed set $\Psi \cup (\mu^n \setminus T^n)$, putting

$$F(\alpha) = a^-$$

for all $\alpha \in \mu^n \setminus T^n$ that do not belong to Ψ . In the same work [2] it is shown that the mapping F can be extended to a mapping

$$F : \mu^n \rightarrow S^n$$

possessing the property that

$$F^{-1}(E^+) = A_1, \quad F^{-1}(E^-) = A_2.$$

We denote the degree of the mapping F by C_1 .

From the construction of the mappings φ_1 and Ff it follows that the points $\varphi_1(x)$ and $F(f(x))$ cannot be diametrically opposite on the sphere S^n for any x ; consequently, by Hurewicz's theorem ⁽³⁾, these mappings are homotopic and their degrees are equal. But the degree of the mapping Ff is equal to $C \cdot C_1$, where C and C_1 are integers and $|C| > 1$, while the degree of the mapping φ_1 is equal to $+1$ or -1 . Thus a contradiction is obtained. Theorem 3 is proved.

From this result, however, one cannot draw a conclusion about the dimension of the set Ξ (the result conjectured here is $\dim \Xi \leq n - 2$); moreover, there exists an example due to K. A. Sitnikov ⁽⁴⁾ of a two-dimensional set that does not divide any ball in E^3 .

Proof of Theorem 4. We shall also prove this theorem by contradiction. Let K be a locally connected continuum belonging to Ξ and having at a point ξ index greater than two. Then, by the Menger-Nöbeling theorem ⁽⁵⁾, it contains at least three simple arcs having no common points except the point ξ . Without loss of generality one may assume that the indicated arcs are segments of length 1; we shall assume the same about their images. Choose in M^3 a sphere S^2 of sufficiently small radius $r < 1$, with center at the point $a = f^{-1}(\xi)$, such that the image of this sphere under the mapping f is at distance from the point ξ not greater than 1.

The sphere S^2 intersects each of the three mentioned segments in one point. Its image under the mapping f will have the same property with respect to the corresponding triple of segments in μ^3 . Let Σ^2 be the sphere in μ^3 of radius 1 with center at the point ξ , and let h be the projection onto the sphere from its center. Then the mapping $hf : S^2 \rightarrow \Sigma^2$ is continuous and has degree C . At the three points—the ends of the chosen segments—this mapping is one-to-one. On the other hand, H. Hopf proved that the number of points of one-to-one correspondence under a continuous mapping of one two-dimensional sphere onto another with degree whose modulus is greater than 1 does not exceed two ⁽¹⁾. We have obtained a contradiction. Theorem 4 is proved.

As was shown by P. S. Urysohn (⁶), every continuum of index not greater than two is a simple arc or a circle. Thus we obtain

Corollary to Theorem 4. *For $n = 3$, every locally connected continuum $K \subset \Xi$ is a simple arc or a circle.*

Moscow Institute of Physics and Technology

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REFERENCES

1. H. Hopf, Math. Ann., **102** (1929).
2. A. P. Savin, DAN, **138**, No. 5, 1029 (1961).
3. V. Hurewicz, G. Wallman, *Dimension Theory*, Moscow, 1948.
4. K. A. Sitnikov, DAN, **94**, 1007 (1954).
5. K. Menger, *Kurventheorie*, Berlin, 1932.
6. P. S. Urysohn, *Works on Topology and Other Fields of Mathematics*, 1, Moscow-Leningrad, 1951, p. 223.

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