

ON THE THEORY OF PERIODIC GROUPS

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Abstract

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MATHEMATICS

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ON THE THEORY OF PERIODIC GROUPS

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1. The question of the existence in groups of infinite proper subgroups is equivalent to the well-known problem of O. Yu. Schmidt on an infinite group with finite proper subgroups. The most substantial results on this question were obtained by S. N. Chernikov ⁽¹⁾ and M. I. Kargaplov ⁽²⁾.

In the present note it is proved that a periodic group of even order, i.e., a group containing involutions, either coincides with the centralizer of one of its involutions, or it has a proper infinite subgroup. Thus the solution of O. Yu. Schmidt's problem is reduced to the analogous problem for a periodic group of odd order, i.e., a group containing no involutions.

2. Theorem. *An infinite periodic group of even order either coincides with the centralizer of one of its involutions, or possesses a proper infinite subgroup with nontrivial center.*

Proof. Let G be an infinite periodic group of even order and let i be one of its involutions. If the centralizer $C_G(i)$ is infinite, then the assertion of the theorem is true.

Let $C_G(i)$ be a finite group. Denote by T the set of all nonidentity strictly real elements of odd order of the group G with respect to the involution i , i.e., such elements $t \in T$ for which the relation $i^{-1}ti = t^{-1}$ holds. The set T is infinite. Indeed, if it were finite, then in the group G there would exist infinitely many strictly real elements of even order with respect to the involution i . But then, obviously, from the finiteness of the centralizer $C_G(i)$ it would follow that in the group G there exists an involution i_1 with infinite centralizer, and in this case the assertion of the theorem is true.

Let a be some element of T . Then $a = ik$, where k is some involution, with $k \neq i$. From what was proved above it follows that in the set T there exists only a finite number of elements for which the orders of elements of the form $u = ib^{-1}kb$, $b \in T$, are even. Removing such elements from T , we again obtain an infinite set M . Let c be some element of M . Consider the element $s = ic^{-1}kc$. Since the order of the element s is odd, for some element s_1 of $\{s\}$ we shall have $s_1^{-1}c^{-1}kcs_1 = i$. On the other hand, for some element $a_1 \in \{a\}$ we have $a_1^{-1}ka_1 = i$. From the last two equalities we obtain $a_1s_1^{-1}c^{-1}kcs_1a_1^{-1} = k$.

Consequently, $cs_1a_1^{-1} = h \in C_G(k)$, $s_1 = c^{-1}ha_1$. Let $g = c^{-1}h$. Then $s_1 = ga_1$. Transform the last equality by means of the involution i : $i^{-1}s_1i = i^{-1}gi i^{-1}a_1i$. In view of the strict reality of the elements s_1 and a_1 , we obtain $s_1^{-1} = a_1^{-1}g^{-1} = i^{-1}gia_1^{-1}$, or $a_1i^{-1}gia_1^{-1} = g^{-1}$. Obviously, the element $j = ia_1^{-1}$ is an involution and $j^{-1}(h^{-1}c)j = (h^{-1}c)^{-1}$. Since c is an arbitrary element of M , we have proved that for any element $c \in M$ there exists an element $h \in C_G(k)$ such that $g = hc$ is a strictly real element with respect to the involution $j = ia_1^{-1}$.

Since the subgroup $C_G(k)$ is finite, one can extract in M an infinite sequence of elements $c_1, c_2, \dots, c_m, \dots$ such that for some-

of the element $r \in C_G(k)$ there will correspond a sequence of strictly real elements $q_1 = rc_1, q_2 = rc_2, \dots, q_m = rc_m, \dots$ with respect to the involution $j = ia_1^{-1}$, $j^{-1}rc_1j = c_1^{-1}r^{-1}$, $j^{-1}rc_mj = c_m^{-1}r^{-1}$, $j^{-1}c_m^{-1}r^{-1}j = rc_m$.

From the first and the last equality we obtain $j^{-1}(c_m^{-1}c_1)j = r(c_m c_1^{-1})r^{-1}$, or $a_1i^{-1}(c_m^{-1}c_1)ia_1^{-1} = r(c_m c_1^{-1})r^{-1}$ ($j = ia_1^{-1}$).

Since the elements c_1 and c_m are strictly real with respect to the involution i , from the last equality we obtain $r^{-1}a_1(c_m c_1^{-1})a_1^{-1}r = c_m c_1^{-1}$.

Obviously, the set of distinct elements of the form $c_m c_1^{-1}$ ($m = 1, 2, \dots$) is infinite and, consequently, the centralizer of the element $d = a_1^{-1}r$ is infinite, while $C_G(d) \neq G$. Indeed, suppose that $C_G(d) = G$. Then the involution k centralizes the element $d = a_1^{-1}r$. Since $r \in C_G(k)$, from the permutability of the elements k and d we obtain $k^{-1}dk = ka_1^{-1}kr = a_1^{-1}r$, or $k^{-1}a_1^{-1}k = a_1^{-1}$. The last equality is impossible, since, by assumption, $a_1^{-1}ka_1 = i \neq k$. The theorem is proved.

Corollary 1. *There is no infinite simple group of even order all of whose proper subgroups are finite.*

Corollary 2. *It suffices to solve O. Yu. Schmidt's problem for periodic groups of odd order.*

Corollary 3. *The question of the existence of an infinite abelian subgroup in arbitrary periodic groups reduces to the analogous question for periodic groups of odd order.*

Since a group of odd order is soluble ⁽³⁾, in view of the results of S. N. Chernikov ⁽¹⁾, the theorem proved implies the previously known result ⁽²⁾:

An infinite locally finite group has a proper infinite abelian subgroup.

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Note: Figure translations are in progress. See original paper for figures.

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