

ON AN ELLIPTIC SYSTEM OF SECOND-ORDER EQUATIONS IN THE PLANE

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Abstract

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MATHEMATICS

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ON AN ELLIPTIC SYSTEM OF SECOND-ORDER EQUATIONS IN THE PLANE

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The second-order differential equation

$$D_{\lambda}(w) \equiv \partial_z^2 w + A(z)\partial_{\bar{z}} w + \lambda B(z)w = h(z) \quad (1)$$

$$(2\partial_{\bar{z}} = \partial_x + i\partial_y, \quad \partial_z^2 \equiv \partial_z(\partial_{\bar{z}})),$$

in which $A(z)$, $B(z)$, $h(z)$ are given complex-valued functions; λ is a certain parameter; $w(z) = u(x, y) + iv(x, y)$ is the unknown function, is equivalent to a certain real linear elliptic system of two partial differential equations of the second order. In particular, when $A(z) \equiv 0$, $\lambda B(z) \equiv 0$, this equation is equivalent to the well-known system of equations of A. V. Bitsadze⁽²⁻⁴⁾. In the present note, we study some properties of equation (1) connected with the formulation of boundary-value problems for it.

1. General representation of solutions. Let G be an open bounded domain in the complex plane $z = x + iy$, bounded by a closed Lyapunov curve Γ , $\bar{G} = G + \Gamma$. We shall assume that $A(z) \in C_{\nu}^1(\bar{G})$, $B(z) \in C_{\nu}(G)$, $h(z) \in C_{\nu}(\bar{G})$, $0 < \nu < 1$. Every solution of equation (1) belonging to the class $C^2(G)$ will be called a regular solution. Let $\lambda_1, \lambda_2, \dots$ ($0 < |\lambda_1| < |\lambda_2| < \dots$) be the eigenvalues of the homogeneous Fredholm-type integral equation

$$\begin{aligned} w(z) + \lambda T(w) &\equiv \\ &\equiv w(z) + \frac{\lambda}{\pi} \iint_G \frac{w^*(\zeta) - w^*(z)}{\zeta - z} e^{w(\zeta)} B(\zeta) w(\zeta) dG_{\zeta} = 0, \end{aligned} \quad (2)$$

where

$$w^*(z) = \frac{1}{\pi} \iint_G \frac{\exp[-w(\zeta)]}{\zeta - z} dG_{\zeta}, \quad w(z) = -\frac{1}{\pi} \iint_G \frac{A(\zeta)}{\zeta - z} dG_{\zeta}.$$

If $\lambda \neq \lambda_k$, then there exist functions $w_1(z)$, $w_2(z)$ which are regular in G solutions of the homogeneous equation (1) ($h(z) \equiv 0$), continuous in the whole plane together with $\partial_{\bar{z}}w_1$, $\partial_{\bar{z}}w_2$, such that $w_1(z)$, $w_2(z)$, $\partial_{\bar{z}}w_1$, $\partial_{\bar{z}}w_2$ are holomorphic outside $G + \Gamma$ and $w_1(\infty) = 1$, $w_2(\infty) = 0$, $\partial_{\bar{z}}w_1|_{z=\infty} = 0$, $\partial_{\bar{z}}w_2|_{z=\infty} = 1$. These functions can be obtained by solving the integral equations $w(z) + \lambda T(w) = 1$, $w(z) + \lambda T(w) = w^*(z)$. It is not difficult to verify that

$$w_1(z) \cdot \partial_{\bar{z}}w_2 - w_2(z) \cdot \partial_{\bar{z}}w_1 \equiv \exp[-w(z)].$$

Let $w(z)$ be an arbitrary regular in G solution of the homogeneous equation (1). Introducing into consideration the functions

$$v_1(z) = -\partial_{\bar{z}}w_1 \cdot w(z) + w_1(z) \cdot \partial_{\bar{z}}w, \quad v_2(z) = \partial_{\bar{z}}w_2 \cdot w(z) - w_2(z) \cdot \partial_{\bar{z}}w,$$

we have

$$\partial_z v_1 + A(z)v_1 = 0, \quad \partial_z v_2 + A(z)v_2 = 0.$$

Hence it follows that

$$\partial_z w_2 \cdot w(z) - w_2(z) \cdot \partial_z w = \varphi(z) \exp[-\omega(z)],$$

$$-\partial_z w_1 \cdot w(z) + w_1(z) \cdot \partial_z w = \psi(z) \exp[-\omega(z)],$$

where $\varphi(z)$, $\psi(z)$ are arbitrary functions holomorphic in G . Solving the latter relations with respect to $w(z)$, we obtain the representation

$$w(z) = w_1(z)\varphi(z) + w_2(z)\psi(z). \quad (3)$$

Formula (3) gives the general representation of all regular solutions of the homogeneous equation (1) for $\lambda \neq \lambda_k$ in terms of two arbitrary holomorphic functions $\varphi(z)$, $\psi(z)$. Adding to (3) a particular solution

$$\frac{1}{\pi} \iint_G \frac{w_1(z)w_2(\zeta) - w_2(z)w_1(\zeta)}{\zeta - z} h(\zeta) \exp \omega(\zeta) dG_\zeta$$

of the nonhomogeneous equation (1), we obtain the formula for the general representation of all regular solutions of equation (1):

$$w(z) = w_1(z)\varphi(z) + w_2(z)\psi(z) + \frac{1}{\pi} \iint_G \frac{w_1(z)w_2(\zeta) - w_2(z)w_1(\zeta)}{\zeta - z} h(\zeta)e^{\omega(\zeta)} dG_\zeta. \quad (4)$$

Formula (4) is valid for $\lambda \neq \lambda_k$. In particular, it is valid for sufficiently small values of $|\lambda|$ ($0 < |\lambda| < |\lambda_1|$).

From representation (3) the following assertion follows easily:

Theorem 1. If $\lambda \neq \lambda_k$ and the angular boundary values of a regular solution $w(z)$ of the homogeneous equation (1) vanish together with $\partial_z w$ on an arc $\gamma \subset \Gamma$, then $w(z) \equiv 0$.

2. The Dirichlet problem. Find regular solutions of equation (1), continuous in $\overline{G} + \Gamma$, satisfying the condition

$$w|_\Gamma = 0. \quad (D)$$

The problem of determining a solution regular in G of the equation

$$D_\lambda^*(v) \equiv \partial_z^2 v - \partial_z(Av) + \lambda B(z)v = 0, \quad (5)$$

continuous in $\overline{G}$, satisfying condition (D), will be called problem (D_0^*) .

From Green's identity

$$\iint_G \{vD_\lambda(w) - wD_\lambda^*(v)\} dx dy = \frac{1}{2i} \int_\Gamma \{v(z)(\partial_z w + A(z)w) - w\partial_z v\} dz$$

it follows that, for the solvability of the Dirichlet problem, it is necessary that the function $h(z)$ satisfy the equality

$$\iint_G h(z)v(z) dx dy = 0 \quad (6)$$

for every regular solution $v(z)$ of problem (D_0^*) . However, this condition, generally speaking, is not sufficient for solvability of the Dirichlet problem. Below we shall indicate one criterion for the sufficiency of condition (6) in terms of the functions $w_1(z)$, $w_2(z)$.

Using representation (4), we reduce the boundary value problem (D) to the following problem of determining holomorphic functions $\varphi(z)$, $\psi(z)$:

$$w_1(t_0)\varphi(t_0) + w_2(t_0)\psi(t_0) = f(t_0), \quad t_0 \in \Gamma, \quad (7)$$

where

$$f(t_0) = -\frac{1}{\pi} \iint_G \frac{w_1(t_0)w_2(\xi) - w_2(t_0)w_1(\xi)}{\xi - t_0} h(\xi) \exp \omega(\xi) dG_\xi.$$

It follows immediately from condition (7) that the Dirichlet problem cannot be nontrivial. Let $|\lambda|$ be sufficiently small ($0 < |\lambda| < |\lambda_1|$). Then $w_1(z) \neq 0$, and condition (7) can be written as follows:

$$\varphi(t_0) + \tilde{w}(t_0)\psi(t_0) = \tilde{f}(t_0), \quad (8)$$

where $\tilde{w}(z) = w_2(z)/w_1(z)$, $\tilde{f}(t_0) = f(t_0)/w_1(t_0)$. Using the fact that $\varphi(t_0)$ is the boundary value of a function holomorphic in G , condition (8) can also be written as

$$\tilde{w}(t_0)\psi(t_0) - \frac{1}{\pi i} \int_\Gamma \frac{\tilde{w}(t)\psi(t) dt}{t - t_0} = F(t_0) = \tilde{f}(t_0) - \frac{1}{\pi i} \int_\Gamma \frac{\tilde{f}(t) dt}{t - t_0}. \quad (9)$$

Representing now the function $\psi(z)$ in the form of a Cauchy-type integral with density $\mu(t)$, determined up to the boundary value of an arbitrary function holomorphic outside $G + \Gamma$ and vanishing at infinity, and using the formula for interchanging singular integrals, we obtain, for determining $\mu(t)$, the integral equation

$$\frac{1}{2\pi i} \int_\Gamma \frac{\tilde{w}(t_0) - \tilde{w}(t)}{t - t_0} \mu(t) dt + \frac{1}{2\pi^2} \int_\Gamma \left\{ \int_\Gamma \frac{\tilde{w}(t) dt}{(t - t_0)(\tau - t)} \right\} \mu(\tau) d\tau = F(t_0). \quad (10)$$

The continuous function $\tilde{w}(t)$ is the boundary value of the function $w_2(z)/w_1(z)$, holomorphic outside $G + \Gamma$, and vanishing at infinity. Therefore

$$\int_\Gamma \frac{\tilde{w}(t) dt}{(t - t_0)(\tau - t)} = \frac{1}{\tau - t_0} \left\{ \int_\Gamma \frac{\tilde{w}(t) dt}{t - t_0} - \int_\Gamma \frac{\tilde{w}(t) dt}{t - \tau} \right\} = \frac{\pi}{i} (\tilde{w}(t_0) - \tilde{w}(\tau)),$$

and equation (10) takes the form

$$K(\mu) \equiv \frac{1}{\pi i} \int_\Gamma \frac{w_1(t)w_2(t_0) - w_2(t)w_1(t_0)}{w_1(t)(t - t_0)} \mu(t) dt = H(t_0) = w_1(t_0)F(t_0). \quad (11)$$

In view of the Hölder continuity of the functions $w_1(t)$ and $w_2(t)$, equation (11) is a Fredholm integral equation of the first kind. From the very method of

deriving equation (11), it follows that it is equivalent to the Dirichlet problem, while the corresponding homogeneous equation $K(\mu) = 0$ has an infinite set of linearly independent solutions.

The condition for normal solvability of equation (11) has the form

$$\int_{\Gamma} H(t)\nu(t) dt = 0, \quad (12)$$

where $\nu(t)$ is any solution of the homogeneous equation $K'(\nu) = 0$ adjoint to it. It is easy to see that if $\nu(t)$ is a solution of the equation $K'_1(\nu) = 0$, then $\nu_1(t) = w_1(t)\nu(t)$ is a solution of the equation $K(\nu) = 0$. Substituting the value of the function $H(t)$ into condition (12), we easily obtain a condition of the form (6), in which the function $v(z)$ has the form

$$v(z) = -\frac{\exp \omega(z)}{\pi} \int_{\Gamma} \frac{w_1(t)w_2(z) - w_2(t)w_1(z)}{w_1(t)(z-t)} \tilde{\nu}(t) dt, \quad (13)$$

where

$$\tilde{\nu}(t) = \nu_1(t) + \frac{1}{\pi i} \int_{\Gamma} \frac{\nu_1(\tau) d\tau}{\tau - t}.$$

But it is easy to verify that $\tilde{\nu}(t)$ satisfies the homogeneous equation $K(v) = 0$. Hence it follows that

$$\lim_{z \rightarrow t_0} v(z) = v(t_0) = -i \exp \omega(t_0) K(\tilde{\nu}) = 0,$$

i.e., the function (13) satisfies condition (D). On the other hand, direct application of the operation $\partial_{\bar{z}} K$ to (13) gives $\mathcal{D}_{\lambda}^*(v) \equiv 0$ in G . Thus, the condition of normal solvability of the integral equation (11) is equivalent to the condition of normal solvability of the Dirichlet problem (6):

Theorem 2. *If $|\lambda|$ is sufficiently small, then the Dirichlet problem is not a Noether problem. It is normally solvable in the sense of Hausdorff if and only if the integral equation (11) is normally solvable in the sense of Hausdorff.*

3. Noether problems. For the equation $\mathcal{D}_{\lambda}(w) = h(z)$ consider the boundary-value problem:

$$\operatorname{Re}\{a(t)w(t) + b(t)\partial_{\bar{t}}w\} = 0, \quad \operatorname{Re}\{c(t)w(t) + d(t)\partial_t w\} = 0. \quad (14)$$

If the Hölder-continuous functions $a(t), b(t), c(t), d(t)$ are such that $\theta(t) = a(t)d(t) - b(t)c(t) \neq 0$ for $t \in \Gamma$, then for an arbitrary $(m+1)$ -connected domain G the following result holds:

Theorem 3. For solvability of problem (14) it is necessary and sufficient that the equality

$$\operatorname{Re} \iint h(z)v(z) dx dy = 0$$

hold for every solution $v(z)$ of the homogeneous adjoint problem to (14), and the index of this problem is equal to $2\kappa - 4(m - 1)$, where $\kappa = (2\pi)^{-1}\{\arg \theta(t)\}_\Gamma$. Here the homogeneous adjoint problem to (14) has the form:

$$\mathcal{D}_\lambda^*(v) = 0 \quad \text{in } G,$$

$$\operatorname{Re} \frac{t'(s)}{\theta(t)} \{[A(t)d(t) - c(t)]v(t) - d(t)\partial_{\bar{t}}v\} = 0,$$

$$\operatorname{Re} \frac{t'(s)}{\theta(t)} \{[a(t) - A(t)b(t)]v(t) + b(t)\partial_{\bar{t}}v\} = 0.$$

If G is the unit disk $|z| < 1$, then in the case $b(t) \equiv c(t) \equiv 0$, $a(t) = d(t) \equiv 1$, problem (14), for all values of λ , except perhaps a discrete set $\{\lambda_i\}$, is always solvable, and the corresponding homogeneous problem has exactly two linearly independent solutions.

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Note: Figure translations are in progress. See original paper for figures.

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