

A LIMIT THEOREM FOR THE NUMBER OF CROSSINGS OF A HIGH LEVEL BY A STATIONARY GAUSSIAN PROCESS

Yu. K. Belyaev

1967

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196701.82847>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 519.214.3

MATHEMATICS

Yu. K. BELYAEV

A LIMIT THEOREM FOR THE NUMBER OF CROSSINGS OF A HIGH LEVEL BY A STATIONARY GAUSSIAN PROCESS

(Presented by Academician A. N. Kolmogorov on 28 V 1966)

Let ξ_t be a stationary Gaussian process, $M\xi_t = 0$, $M\xi_s\xi_{s+t} = \rho(t)$, $-\infty < t < +\infty$. It is said that the process ξ_t crosses the level u from below upward at the time τ if $\xi_\tau = u$, and in a sufficiently small neighborhood $(\tau - \varepsilon, \tau + \varepsilon)$ of the point τ , $\varepsilon = \varepsilon(\omega)$ being a random variable, $\xi_s < u$, $\tau - \varepsilon < s < \tau$, $\xi_s > u$, $\tau < s < \tau + \varepsilon$. Denote by $\eta_u(\Delta)$ the number of crossings from below upward of the level u on the interval Δ . Denote by $\tilde{\eta}_u(\Delta)$ the accompanying random variable, taking integer values $\tilde{\eta}_u(\Delta) = \eta_u(\Delta)$ if in the interval Δ there are no crossings following one another after a time less than τ ; otherwise we put $\tilde{\eta}_u(\Delta) = 0$.

Everywhere below it is assumed that the correlation function $\rho(t)$ is twice differentiable and satisfies the conditions $\rho(0) = 1$,

$$|\rho''(t) - \rho''(0)| \leq c/|\ln |t||^{1+\varepsilon}, \quad c, \varepsilon > 0, \quad t \rightarrow 0; \quad (1)$$

$$\rho(t) = o(1/\ln t), \quad \rho'(t) = o(1/\sqrt{\ln t}), \quad t \rightarrow \infty. \quad (2)$$

As the level u is increased ($u \rightarrow \infty$), we shall increase $|\Delta|$ —the length of the interval Δ —in such a way that $e^{-u^2/2}|\Delta| = \text{const}$. For brevity we denote such a coordinated increase of u and $|\Delta|$ by $(u, |\Delta|) \rightarrow \infty$. Denote by $A_u(|\Delta|, \tau)$ the event consisting in the fact that on the interval Δ there are crossings of the level u from below upward which follow one another after a time less than τ .

Theorem 1. *If the correlation function of the Gaussian process ξ_t satisfies (1), then for $\tau \leq v_u = \exp\{o(u^2)\}$, $u \rightarrow \infty$,*

$$\lim_{(u, |\Delta|) \rightarrow \infty} \mathbf{P}\{A_u(|\Delta|, \tau)\} = 0. \quad (3)$$

In the proof of the theorem the limiting relation

$$\lim_{u \rightarrow \infty} \frac{1}{\tau} e^{u^2/2} \int_0^{2\tau} \int_0^{2\tau} \mathbf{M}(\dot{\xi}_{t_1}^+ \dot{\xi}_{t_2}^+ \mid \xi_{t_j} = u, j = 1, 2) p_{t_1, t_2}(u, u) dt_1 dt_2 = 0,$$

is used, where $p_{t_1 t_2}(x_1, x_2)$ is the joint probability density of the values ξ_{t_1}, ξ_{t_2} , and $\dot{\xi}_t^+ = \dot{\xi}_t$ if $\dot{\xi}_t > 0$; $\dot{\xi}_t^+ = 0$ if $\dot{\xi}_t \leq 0$.

It follows from (3) that, for $(u, |\Delta|) \rightarrow \infty$, the limiting distributions of $\eta_u(\Delta)$ and $\tilde{\eta}_u(\Delta)$ coincide.

Consider the k -fold integrals defined by the formulas

$$J_{(k)}(\Delta; u, \tau) = \int_{\substack{t_i \in \Delta, \\ i, j=1, \dots, k}} \dots \int \mathbf{M} \left(\prod_{i=1}^k \dot{\xi}_{t_i}^+ \mid \xi_{t_j} = u, j = 1, \dots, k \right) p_{t_1 \dots t_k}(u \dots u) dt_1 \dots dt_k. \quad (4)$$

It is easy to show that under the assumptions made $J_{(k)}(\Delta; u, \tau) < \infty$. For the proof of the main assertions the following lemma is useful.

Lemma 1. Under a coordinated increase

$$(u, |\Delta|) \rightarrow \infty, \quad |\Delta| = \frac{2\pi\mu}{\sqrt{-\rho''(0)}} e^{u^2/2}$$

there exist, for every k , numbers $\tau_k > 0$ such that for all $\tau, \tau_k < \tau < \sqrt{|\Delta|}$,

$$\lim_{(u, |\Delta|) \rightarrow \infty} J_{(k)}(\Delta; u, \tau) = \mu^k, \quad k = 1, 2, \dots. \quad (5)$$

Let us now consider the behavior of the factorial moments $\tilde{J}_k(\Delta, u)$ for the random variables $\tilde{\eta}_u(\Delta)$; recall that

$$\tilde{J}_{(k)}(\Delta, u) = \mathbf{M} \tilde{\eta}_u(\Delta) [\tilde{\eta}_u(\Delta) - 1] \dots [\tilde{\eta}_u(\Delta) - k + 1].$$

Theorem 2. Under a coordinated increase $(u, |\Delta|) \rightarrow \infty$,

$$|\Delta| = \frac{2\pi\mu}{\sqrt{-\rho''(0)}} e^{u^2/2},$$

and assuming conditions (1), (2) are satisfied,

$$\lim_{(u, |\Delta|) \rightarrow \infty} \tilde{J}_{(k)}(\Delta, u) = \mu^k, \quad k = 1, 2, \dots. \quad (6)$$

In the proof of Theorem 2, assertion (5) is used, as well as the fact that the limiting behavior of $\tilde{J}_{(k)}$ and $J_{(k)}$ is the same. Here the method used in the proof

of Theorem 1 of the author's paper ⁽¹⁾ proves useful. The main result of the paper is obtained as a consequence of the theorem on convergence of moments ⁽²⁾, p. 198).

Theorem 3. If, for the correlation function $\rho(t)$ of a stationary Gaussian process, (1), (2) are satisfied, then under a coordinated increase $(u, |\Delta|) \rightarrow \infty$,

$$\lim_{(u, |\Delta|) \rightarrow \infty} \mathbf{P}\{\eta_u(\Delta) = k\} = \frac{(\mu|\Delta|)^k}{k!} e^{-\mu|\Delta|}, \quad k = 0, 1, \dots$$

It can be shown that the joint multidimensional distributions will also be Poisson. Theorem 3 is a generalization of Cramér's result ⁽³⁾, in which the existence of four derivatives of the correlation function $\rho(t)$ and $\rho(t) = o(t^{-\varepsilon})$, $t \rightarrow \infty$, $\varepsilon > 0$, are assumed. Using Theorem 3, one can obtain a generalization of one more result of Cramér ⁽⁴⁾.

Theorem 4. If conditions (1), (2) are satisfied, then

$$\lim_{T \rightarrow \infty} \mathbf{P}\left\{\max_{0 \leq t \leq T} \xi_t \leq \sqrt{2 \ln T} + \frac{z - A}{\sqrt{2 \ln T}}\right\} = e^{-e^{-z}}, \quad A = \ln \frac{2\pi}{\sqrt{-\rho''(0)}}.$$

Moscow State University
named after M. V. Lomonosov

Received
19 V 1966

REFERENCES

1. Yu. K. Belyaev, *Theory of Probability and Its Applications*, **11**, 1, 120 (1966).
2. M. Loeve, *Probability Theory*, IL, 1962.
3. H. Cramér, Ark. Matem., **6**, 20, 337 (1966).
4. H. Cramér, *Theory of Probability and Its Applications*, **10**, 1, 137 (1965).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.