

ON THE DESCRIPTION OF GENERALIZED EXTENSIONS WITH ONE-DIMENSIONAL IMAGINARY COMPONENT OF THE DIFFERENTIATION OPERATOR WITHOUT SPECTRUM

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Abstract

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MATHEMATICS

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ON THE DESCRIPTION OF GENERALIZED EXTENSIONS WITH ONE-DIMENSIONAL IMAGINARY COMPONENT OF THE DIFFERENTIATION OPERATOR WITHOUT SPECTRUM

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I. Let T be a closed operator acting in a Hilbert space $\mathcal{G}$, for which

$$T_0 \subset T, \quad T^* \subset T_0^*, \quad (1)$$

where T_0 is a symmetric operator with dense domain. Consider the Hilbert space $\mathcal{G}_+ = D_{T_0^*}$ with scalar product

$$(f, g)_+ = (T_0^* f, T_0^* g)_0 + (f, g)_0 \quad (f, g \in \mathcal{G}_+). \quad (2)$$

Construct a triple of spaces $\mathcal{G}_+ \subseteq \mathcal{G}_0 \subseteq \mathcal{G}_-$, where $\mathcal{G}_-$ is the space of generalized elements generating antilinear functionals on $\mathcal{G}_+$ (see ^(2,4)).

In ^(4,5) it was shown that T and T^* can always be extended to $\mathcal{G}_+ = D_{T_0^*}$ in such a way that the resulting extensions $T_{\mathcal{G}_+}$ and $T_{\mathcal{G}_+}^*$ are adjoint to one another in the generalized sense; moreover, $T_{\mathcal{G}_+}$ was called a generalized extension of the operator T .

M. S. Livšic is responsible for ⁽¹⁾ the well-known theorem that every quasidissipative extension without spectrum of a simple symmetric operator with deficiency index $(1, 1)$ is unitarily equivalent to the differentiation operator T

$$\begin{aligned} Tf &= \frac{1}{i} \frac{df}{dx} & (f(0) = 0, \ 0 \leq x \leq \eta), \\ T^* f &= \frac{1}{i} \frac{df}{dx} & (f(\eta) = 0, \ 0 \leq x \leq \eta). \end{aligned} \quad (3)$$

It is therefore natural that it is of interest to give a description of all generalized extensions of the differentiation operator (3).

In the present paper we give a description of the generalized extensions $T_{\mathcal{G}_+}$ of this operator that have a one-dimensional imaginary component $(T_{\mathcal{G}_+} - T_{\mathcal{G}_+}^\times)/i$, which makes it possible to single out in a natural way an important class of such extensions.

II. Let $\mathcal{G}_0 = L_2[0, \eta]$, and let

$$T_0 f = \frac{1}{i} \frac{df}{dx} \quad (f(0) = f(\eta) = 0).$$

It is known ⁽³⁾ that $T_0^* f = \frac{1}{i} \frac{df}{dx}$, and $D_{T_0^*}$ consists of absolutely continuous functions on $[0, \eta]$ whose derivatives belong to $L_2[0, \eta]$. Obviously, the operators T and T^* satisfy condition (1). It follows from (2) that $\mathcal{G}_+ = W_2^{(1)}[0, \eta] = D_{T_0^*}$, and the scalar product in $\mathcal{G}_+$ is given by the formula

$$(f, g)_+ = \int_0^\eta f'(x) \overline{g'(x)} dx + \int_0^\eta f(x) \overline{g(x)} dx.$$

Let us construct the space $\mathcal{G}_-$, so that $\mathcal{G}_+ \subseteq \mathcal{G}_0 \subseteq \mathcal{G}_-$.

Consider the family of operators acting from $\mathcal{G}_+$ into $\mathcal{G}_-$:

$$T_{G_+}(\xi)f = \frac{1}{i} \frac{df}{dx} + if(0)[e^{i\xi\eta}\delta(x-\eta) - \delta(x)], \quad (4)$$

$$\left(\begin{array}{c} -\infty < \xi < +\infty \\ 0 \leq x \leq \eta \end{array} \right)$$

$$T_{G_+}^\times(\xi)f = \frac{1}{i} \frac{df}{dx} + if(\eta)e^{-i\xi\eta}[e^{i\xi\eta}\delta(x-\eta) - \delta(x)]; \quad (5)$$

here $\delta(x-\eta)$, $\delta(x)$ are generalized elements (delta functions) generating the functionals

$$(\delta(x-\eta), f)_0 = \overline{f(\eta)}, \quad (\delta(x), f)_0 = \overline{f(0)}.$$

It is not hard to see that for each fixed ξ the operators $T_{G_+}(\xi)$ and $T_{G_+}^\times(\xi)$ are extensions of T and T^* , respectively, and moreover

$$(T_{G_+}(\xi)f, g)_0 = (f, T_{G_+}^\times(\xi)g)_0 \quad (f, g \in \mathcal{G}_+).$$

In addition, it follows from (4), (5) that

$$\frac{T_{G_+}(\xi) - T_{G_+}^\times(\xi)}{i} f = (f, [e^{i\xi\eta}\delta(x-\eta) - \delta(x)])_0 J [e^{i\xi\eta}\delta(x-\eta) - \delta(x)]$$

$$(J = -1). \quad (6)$$

Relation (6) shows that for every ξ the imaginary component of the family of operators $T_{G_+}(\xi)$ is one-dimensional.

We shall show that the family of operators (4) exhausts all generalized extensions of the operator T on $\mathcal{G}_+ = D_{T_0^*}$ that have a one-dimensional imaginary component.

III. In what follows we shall need one theorem, whose proof may be found in ⁽⁵⁾ and whose statement we give for convenience.

Theorem 1. Let the operator T satisfy conditions (1). In order that the operators T and T^* be extendable to the whole space

$$\mathcal{G}_+ = D_{T_0^*} = D_T + D_{T^*}$$

in such a way that the resulting extensions T_{G_+} and $T_{G_+}^\times$ are adjoint to each other in the generalized sense, it is necessary and sufficient that there exist linear operators $P(D_{T^*} \rightarrow \mathcal{G}_-)$ and $Q(D_T \rightarrow \mathcal{G}_-)$ possessing the following properties:

$$\begin{aligned} 1) \quad & (Pf_2, g_2)_0 = (f_2, T^*g_2)_0, \quad (Qf_1, g_1)_0 = (f_1, Tg_1)_0 \\ & (f_1, g_1 \in D_T, f_2, g_2 \in D_{T^*}); \\ 2) \quad & (Pf_2, g_1)_0 = (f_2, Qg_1)_0 \quad (g_1 \in D_T, f_2 \in D_{T^*}); \\ 3) \quad & P\varphi = T\varphi, \quad Q\varphi = T^*\varphi \quad (\varphi \in D_{T_0} = D_T \cap D_{T^*}). \end{aligned} \quad (7)$$

Moreover,

$$\begin{aligned} T_{G_+}f &= Tf_1 + Pf_2, \\ T_{G_+}^\times f &= Qf_1 + T^*f_2 \end{aligned} \quad (f = f_1 + f_2, f_1 \in D_{T^*}, f \in \mathcal{G}_+). \quad (8)$$

Theorem 2. There do not exist two distinct generalized extensions T_{G_+} and T_{G_+}' of the operator

$$Tf = \frac{1}{i} \frac{df}{dx} \quad (f(0) = 0, 0 \leq x \leq \eta)$$

on $\mathcal{G}_+ = D_{T_0^*}$, for which the real parts would be extensions of the self-adjoint operator

$$A_0\psi = \frac{1}{i} \frac{d\psi}{dx} \quad (\psi(0) + \theta\psi(\eta) = 0, \quad |\theta| = 1).$$

Proof. Let

$$T_{G_+} = A + iB, \quad T'_{G_+} = A' + iB', \quad (9)$$

where $A = (T_{G_+} + T_{G_+}^\times)/2$, $A' = (T'_{G_+} + T_{G_+}^{\prime\,\times})/2$, etc.

From theorem (1) it follows that

$$\begin{aligned} T_{\mathfrak{G}_+} f &= T f_1 + P f_2, & T'_{\mathfrak{G}_+} f &= T f_1 + P' f_2, \\ T_{\mathfrak{G}_+}^\times f &= Q f_1 + T^* f_2, & T_{\mathfrak{G}_+}^{\prime\,\times} f &= Q' f_1 + T^* f_2. \end{aligned} \quad (10)$$

From (9) and (10) it follows that

$$\begin{aligned} A f &= \frac{1}{2}(T f_1 + T^* f_2 + Q f_1 + P f_2), \\ (f &= f_1 + f_2, \quad f_1 \in D_T, \quad f_2 \in D_{T^*}), \\ A' f &= \frac{1}{2}(T f_1 + T^* f_2 + Q' f_1 + P' f_2). \end{aligned} \quad (11)$$

In (5) it was shown that $T_0^* f = T f_1 + T^* f_2$ ($f = f_1 + f_2$, $f_1 \in D_T$, $f_2 \in D_{T^*}$). Therefore, taking (11) into account, we obtain

$$A f = \frac{1}{2}(T_0^* f + Q f_1 + P f_2), \quad A' f = \frac{1}{2}(T_0^* f + Q' f_1 + P' f_2). \quad (12)$$

Suppose now that A and A' are extensions of the self-adjoint operator A_0 , i.e. $A f_0 = A_0 f_0$, $A' f_0 = A_0 f_0$ ($f_0 \in D_{A_0}$). Since $T_0 \subset A_0 \subset T_0^*$, we have

$$A_0 f_0 = \frac{1}{2}(A_0 f_0 + Q f_1^0 + P f_2^0),$$

$$A_0 f_0 = \frac{1}{2}(A_0 f_0 + Q' f_1^0 + P' f_2^0), \quad (f_0 = f_1^0 + f_2^0, \quad f_1^0 \in D_T, \quad f_2^0 \in D_{T^*}, \quad f_0 \in D_{A_0}).$$

From the last relations it follows that

$$(Q - Q') f_1^0 = -(P - P') f_2^0. \quad (13)$$

From equalities (7) and (13) it follows that

$$((Q - Q')f_1^0, g_1)_0 = -((P - P')f_2^0, g_1)_0 = 0,$$

$$((P - P')f_2^0, g_2)_0 = -((Q - Q')f_1^0, g_2)_0 = 0.$$

Hence

$$((Q - Q')f_1^0, g_1)_0 = 0, \quad ((Q - Q')f_1^0, g_2)_0 = 0,$$

therefore,

$$((Q - Q')f_1^0, g)_0 = 0 \quad (g = g_1 + g_2, \quad g_1 \in D_T, \quad g_2 \in D_{T^*})$$

and, consequently, $(Q - Q')f_1^0 = 0$. Similarly, $(P - P')f_2^0 = 0$. From (8) and (12) it follows that $T_{\mathfrak{G}_+}f_0 = T'_{\mathfrak{G}_+}f_0$, $Af_0 = A'f_0$.

Thus, we have proved that the generalized extensions $T_{\mathfrak{G}_+}$ and $T'_{\mathfrak{G}_+}$ of the operator T coincide on the subspace D_{A_0} under the condition that their real parts are extensions of the self-adjoint operator A_0 . We shall now show that $T_{\mathfrak{G}_+}$ and $T'_{\mathfrak{G}_+}$ coincide on the whole space $\mathfrak{G}_+$.

Let f_1^0 be an arbitrary function from D_T . Then, as is easy to see, one can choose a function $f_2^0 \in D_{T^*}$ such that $f_0 = f_1^0 + f_2^0$ belongs to the set D_{A_0} . From (9) it follows that $T_{\mathfrak{G}_+}^\times = A - iB$, $T_{\mathfrak{G}_+}'^\times = A' - iB'$. Therefore

$$\begin{aligned} Tf_1^0 &= Af_1^0 + iBf_1^0, & Tf_1^0 &= A'f_1^0 + iB'f_1^0, \\ T^*f_2^0 &= A'f_2^0 - iB'f_2^0, & T^*f_2^0 &= Af_2^0 - iBf_2^0. \end{aligned} \quad (14)$$

From relations (14) it follows that

$$Tf_1^0 + T^*f_2^0 = A(f_1^0 + f_2^0) + iB(f_1^0 - f_2^0),$$

$$Tf_1^0 + T^*f_2^0 = A'(f_1^0 + f_2^0) + iB'(f_1^0 - f_2^0);$$

whence it follows immediately that

$$(A - A')f_0 + i(B - B')(f_1^0 - f_2^0) = 0 \quad (f_0 = f_1^0 + f_2^0, \quad f_0 \in D_{A_0}). \quad (15)$$

Since, as we have shown, $T_{\mathfrak{g}_+} f_0 = T'_{\mathfrak{g}_+} f_0$, it already follows easily that $Bf_0 = B'f_0$, $Af_0 = A'f_0$. Therefore from (15) $(B - B')(f_1^0 - f_2^0) = 0$, i.e.,

$$(B - B')f_1^0 = (B - B')f_2^0. \quad (16)$$

Since $f_1^0 + f_2^0 = f_0$, we have

$$(B - B')f_1^0 + (B - B')f_2^0 = 0. \quad (17)$$

From (16) and (17) we obtain that $(B - B')f_1^0 = 0$ for an arbitrary function $f_1^0 \in D_T$. Similarly one proves that $(B - B')f_2^0 = 0$ for an arbitrary function $f_2^0 \in D_{T^*}$. Hence it follows that $(B - B')f = 0$ ($f \in \mathfrak{G}_+$). Since, when the imaginary components coincide, $B = B'$, the generalized extensions $T_{\mathfrak{g}_+}$ and $T'_{\mathfrak{g}_+}$, as is known (5), also coincide, it follows that $T_{\mathfrak{g}_+} = T'_{\mathfrak{g}_+}$. The theorem is proved.

Theorem 3. In order that a generalized extension $T_{\mathfrak{g}_+}$ of the operator

$$Tf = \frac{1}{i} \frac{df}{dx} \quad (f(0) = 0, \ 0 \leq x \leq \eta)$$

have a one-dimensional imaginary component, it is necessary and sufficient that the real part $(T_{\mathfrak{g}_+} + T_{\mathfrak{g}_+}^\times)/2$ be an extension of some self-adjoint operator A_0 which is an extension of the operator

$$T_0f = \frac{1}{i} \frac{df}{dx} \quad (f(0) = f(\eta) = 0, \ 0 \leq x \leq \eta).$$

Proof. It is known (3) that all self-adjoint extensions of the operator T_0 have the form

$$T(\theta)f = \frac{1}{i} \frac{df}{dx} \quad (f(0) + \theta f(\eta) = 0, \ |\theta| = 1, \ 0 \leq x \leq \eta). \quad (18)$$

Let now $T_{\mathfrak{g}_+}$ be some generalized extension of the operator T , for which $(T_{\mathfrak{g}_+} + T_{\mathfrak{g}_+}^\times)/2$ is an extension of the operator $T(\theta)$ for fixed θ ($|\theta| = 1$). From (4) it follows that ξ can be chosen so that $(T_{\mathfrak{g}_+}(\xi) + T_{\mathfrak{g}_+}^\times(\xi))/2$ will also be an extension of $T(\theta)$ for the same θ . By virtue of Theorem 2, $T_{\mathfrak{g}_+}$ and $T_{\mathfrak{g}_+}^\times(\xi)$ must coincide. From (6) it follows that $T_{\mathfrak{g}_+}$ has a one-dimensional imaginary component. Sufficiency is proved.

Let now $T_{\mathfrak{g}_+}$ be a generalized extension of the operator T having a one-dimensional imaginary component, i.e.,

$$T_{\mathfrak{g}_+} = A + i(\cdot, e_0)_0 J e_0 \quad (A = A^\times, \ J = \pm 1). \quad (19)$$

From (19) it follows that

$$Tf_1 = Af_1 + i(f_1, e_0)_0 Je_0, \quad T^*f_2 = Af_2 - i(f_2, e_0)_0 Je_0, \quad (f_1 \in D_T, f_2 \in D_{T^*}),$$

hence

$$Af = T_0^*f - i(f_1 - f_2, e_0)_0 Je_0 \quad (f = f_1 + f_2, f_1 \in D_T, f_2 \in D_{T^*}). \quad (20)$$

It is not difficult to show that for any $f_1 \in D_T$ one can indicate such an $f_2 \in D_{T^*}$ that $(f_1, e_0)_0 = (f_2, e_0)_0$. Therefore the operator A , as is seen from (20), will act on vectors $f = f_1 + f_2$ ($(f_1, e_0)_0 = (f_2, e_0)_0$) as an operator in the space $\mathfrak{G}_0$ and will be a symmetric extension of T_0 . Consequently, as is known (3), it must coincide with one of the operators $T(\theta)$ (see (18)). The theorem is proved.

Corollary. The totality of generalized extensions with one-dimensional imaginary component of the differentiation operator T (3) consists of the family (4).

Let us note in conclusion that all generalized extensions (4) in the space $\mathfrak{G}$ also have no spectrum in the finite part of the plane (see (5)).

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Note: Figure translations are in progress. See original paper for figures.

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