

ON PERFECT MAPS AND CONDENSATIONS

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Abstract

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MATHEMATICS

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ON PERFECT MAPS AND CONDENSATIONS

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In (1) the following assertions were announced (without proofs):

- I. If a space is mapped perfectly onto a space of weight τ and is condensed onto a space of weight τ (in general, another one), then its own weight does not exceed τ .
- II. If a space is mapped perfectly onto a space with a countable network and if it is condensed onto (in general, another) space with a countable network, then it itself has a countable network*.
- III. If a space can be mapped perfectly onto a symmetrizable space and can be condensed onto a symmetrizable space, then it itself is symmetrizable.
- IV. If a space X can be mapped perfectly onto a space with a refining sequence of covers and if X can be condensed onto a space with a refining sequence of covers, then it itself has a refining sequence of covers.

Still earlier, in (3), it was proved that:

- V. If a perfect preimage of a metric space is condensed onto some metric space, then it is metrizable.

These results were the subject of my report at the International Mathematical Congress in Moscow in 1966. At that time, despite the common scheme of the formulations, it was necessary to give independent proofs. Now a general idea has been found which underlies all five assertions, and a general theorem embracing them.

Definition 1. We shall call a property $\mathcal{P}$ **Hollandian** if both of the following conditions are fulfilled:

- 1) From the fact that $\mathcal{P}$ is possessed by any two topological spaces, it follows that $\mathcal{P}$ is possessed by their topological product.
- 2) If a space has the property $\mathcal{P}$, then every closed subspace of it also has the property $\mathcal{P}$.

Often we shall identify a property with the class of all spaces possessing it (in doing so, of course, properties of equal extent are identified).

In what follows the symbol $\mathcal{P}$ may denote both a certain property and the corresponding class of spaces. Moreover, any class of topological spaces may be regarded as a certain property (namely, the property of belonging to it!). A class of spaces is called Hollandian if the property corresponding to it is Hollandian, i.e., if: 1) together with any two of its elements it also contains their product; 2) together with any one of its elements, every closed subspace of the latter belongs to the given class.

A representative list of perfect properties contains the following obvious

* Definitions of all not entirely standard concepts that are discussed in this paper can be found in (1).

** $\mathcal{P}$ is any topological invariant.

Proposition 1. Each of the classes of spaces listed below is hereditary:

- 1) The class of $\aleph_0$ -spaces (see (5)).
 - 2) The class of completely regular spaces.
 - 3) The class of spaces with the first axiom of countability.
 - 4) The class of spaces whose character at each point does not exceed a cardinal number.
 - 5) The class of spaces of weight $\leq \tau$.
 - 6) The class of continuous images of spaces of weight $\leq \tau$.
 - 7) The class of metrizable spaces.
 - 8) The class of spaces with a refining set of covers.
 - 9) The class of spaces with a uniform base.
 - 10) The class of finite-dimensional spaces.
 - 11) The class of weakly countable-dimensional spaces.
 - 12) The class of countable-dimensional spaces.
 - 13) The class of zero-dimensional spaces.
- (In items 10)–13) any of the dimensions ind and dim is meant.)

- 14) The class of spaces with a point-countable base.
- 15) The class of all spaces representable as the union of a countable set of metrizable subspaces.

Main theorem. If a topological space X can be perfectly mapped onto some space Y with property $\mathfrak{P}$ and condensed onto (most likely another) Hausdorff space Z with property $\mathfrak{P}$, and the property $\mathfrak{P}$ is hereditary, then $\mathfrak{P}$ is inherent also in the space X .

The theorem will follow from Lemmas 1, 2, and 3, to which we shall preface a terminological remark.

Let f and g be mappings of a topological space X into topological spaces Y and Z . The following well-known device makes it possible, from f and g , to define a new mapping $\varphi : X \rightarrow Y \times Z$. Namely, we put $\varphi(x) = (f(x), g(x))$, $x \in X$. The mapping φ will be called the **diagonal product of the mappings** f and g ; we may write in this case $\varphi = D(f, g)$.

Lemma 1. If f is a perfect mapping and g is a continuous mapping, and the image under g is a Hausdorff space, then $\varphi = D(f, g)$ is also a perfect mapping.*

Proof. It is clear that the diagonal product of continuous mappings is a continuous mapping. Therefore φ is continuous. If $t \in \varphi(X)$, then $t = (y, z)$, where $y \in Y$ and $z \in Z$; evidently, then

$$\varphi^{-1}(t) = f^{-1}(y) \cap g^{-1}(z).$$

Consequently, $\varphi^{-1}(t)$ is bicomact for every $t \in \varphi(X)$. Let us prove that $\varphi : X \rightarrow \varphi(X)$ is a closed mapping.

Let P be any closed subset of the space X and let $x_0 \in X \setminus P$. We shall show (this is sufficient) that either 1) the point $t_0 = \varphi(x_0) = (f(x_0), g(x_0))$ has a neighborhood in $Y \times Z$ not intersecting the set $\varphi(P)$, or 2) $t_0 \in \varphi(P)$. Put $y_0 = f(x_0)$, $z_0 = g(x_0)$, $\Phi_0 = f^{-1}(y_0)$, $F_0 = \Phi_0 \cap P$, and $B_0 = g(F_0)$. Two cases are possible: I $g(x_0) \notin B_0$ and II $g(x_0) \in B_0$. If II holds, then $g(x_0) = g(x_1)$ for some $x_1 \in F_0$. Then

$$\varphi(x_0) = (f(x_0), g(x_0)) = (f(x_1), g(x_1)) = \varphi(x_1) \in \varphi P.$$

Thus case 2) has occurred. Suppose now that condition I holds: $g(x_0) \notin B_0$. There exist open, disjoint sets V_0 and G_0 in Y such that $g(x_0) \in V_0$ and $B_0 \subset G_0$. Put $U = f^{-1}G \cup (X \setminus P)$. The set U is open in X and $f^{-1}y_0 \subset U$. By the closedness of the mapping f , there exists an open neighborhood A of the point y_0 in the space—

* This lemma was known for the case when all spaces are completely regular (see (2), Lemma 1.5). It was used by V. I. Ponomarev in the study of spaces

coabsolute with metrizable ones, and was derived by him from a related lemma of V. L. Klyushin (8).

in Y , for which $f^{-1}A \subset U$. Consider the set $O = A \times V_0$. This is an open subset of $Y \times Z$, and

$$\varphi(x_0) = (f(x_0), g(x_0)) = (y_0, g(x_0)) \in A \times V_0 = O.$$

We shall show that $O \cap \varphi(P) = \Lambda$. Choose a point $x \in P$ arbitrarily. If $\varphi(x) \in O$, then $f(x) \in A$ and $g(x) \in V_0$. From the second relation we conclude that $x \notin U$. From the first relation it follows that $x \in U$. We have obtained a contradiction, completing the proof of Lemma 1.

Remark. It is clear that if f is an irreducible mapping, then $\varphi = D(f, g)$ is irreducible.

Definition 2. Mappings $f : X \rightarrow Y$ and $g : X \rightarrow Z$ are called **orthogonal** if, for any $y \in Y$ and $z \in Z$, the set

$$f^{-1}y \cap g^{-1}z$$

is either empty or consists of a single point.

It is clear that the diagonal product $\varphi = D(f, g)$ of orthogonal mappings f and g is one-to-one. Therefore Lemma 1 implies

Lemma 2. *The diagonal product of a perfect mapping and any continuous mapping orthogonal to it is a homeomorphism into.*

Lemma 3. *Let $f : X \rightarrow Y$ be a perfect mapping, $g : X \rightarrow Z$ a continuous mapping, $f(X) = Y$, $g(X) = Z$, and let Z be a Hausdorff space. Then the set*

$$\varphi(X) = \{(f(x), g(x)) : x \in X\}$$

*is closed in the space $Y \times Z$. **

Proof. Let $(y, z) \notin \varphi(X)$, $y \in Y$, $z \in Z$. This means that

$$f^{-1}y \cap g^{-1}z = \Lambda.$$

Then $z \notin gf^{-1}y$, and $gf^{-1}y$ is bicomact. Since Z is a Hausdorff space, there exist disjoint open subsets U and V of the space Z such that $U \ni z$, $V \supset gf^{-1}y$. Put $G = g^{-1}V$; the set G is open in X and $f^{-1}y \subset G$. By the closedness of the mapping f , there exists a neighborhood O of the point y in Y for which $f^{-1}O \subset G$. Then

$$g(f^{-1}O) \subset gg^{-1}V = V,$$

whence it follows that

$$U \cap g(f^{-1}O) = \Lambda,$$

i.e.,

$$g^{-1}U \cap f^{-1}O = \Lambda.$$

Thus $O \times U$ is a neighborhood of the point (y, z) in $Y \times Z$ which does not meet $\varphi(X)$. The lemma is proved.

It is evident that a condensation is orthogonal to any mapping. This observation, in view of Lemmas 2 and 3, completes the proof of the main theorem.

Proposition 1 indicates a number of concrete consequences of the main theorem. These include, in particular, the assertions I, II, IV, V formulated at the beginning of the paper; however, assertion III requires an additional argument, since symmetrizability apparently is not a hereditary property. **

Thus, let $f : X \rightarrow Y$ be a perfect mapping, where Y is a symmetrizable space, and let $g : X \rightarrow Z$ be a condensation, where Z is a symmetrizable space satisfying the Hausdorff separation axiom. Then, by Lemma 2,

$$\varphi : X \rightarrow Y \times Z,$$

where $\varphi = D(f, g)$, is a homeomorphism into. Every symmetrizable space is a k -space (see ⁽¹⁾). Since the inverse image of a k -space under a perfect mapping is a k -space (see ⁽⁴⁾), X is a k -space. Consequently, $\varphi(X)$ is also a k -space. But it is easily verified that if a subspace of a product of symmetrizable spaces is a k -space, then the symmetrization of the product induced on it agrees with the topology, i.e. this subspace is symmetrizable. Thus assertion V is now proved as well. It could also have been proved by a direct construction of a symmetrization, but we wished to show that the technique developed is applicable not only when the property under consideration is hereditary.

Remark. In Proposition 1 only those properties were listed which, when used as the property $\mathcal{P}$ in the formulation of the main theorem, give nontrivial assertions. Therefore, in particular, the class of Hausdorff spaces is not named in the indicated list.

* Here φ , as above, is the diagonal product of the mappings f and g .

** Recall that not all symmetrizable spaces satisfy the first axiom of countability.

On the other hand, a perfect preimage of a perfectly regular space need not be a perfectly regular space (even if both spaces are Hausdorff—see ⁽⁶⁾, 4.1). This means that variant 2 of the main theorem is substantial (Proposition 1). On the other hand, regularity in the class of Hausdorff spaces is a two-sided invariant with respect to perfect mappings (see ⁽⁶⁾).

A curious circumstance: only in items 10–13 of Proposition 1 is there discussion of properties of spaces that are not hereditary with respect to all subspaces (dimension dim). Apparently, almost all known properties of topological spaces that are finitely multiplicative (together with a finite family of factors they belong also to their product) and are hereditary not with respect to arbitrary, but only to closed, subspaces, are necessarily formulated in terms of covers or in terms of an arrangement in an enclosing bicomactum, and belong to

all bicompacta. And such properties, as was proved by van der Slot ⁽⁷⁾, are preserved under one passage to the preimage under a perfect mapping and, consequently, do not fall within the sphere of nontrivial action of the main theorem. It would be interesting to know whether the reasoning just carried out can be made rigorous.

Let us record in conclusion a consequence of variant 13 of the main theorem.

Corollary 1. *If a perfectly disconnected $*$ space can be perfectly mapped onto a zero-dimensional (in the sense of ind or dim) space, then it itself is zero-dimensional (in the sense of ind or dim).*

Moreover, Lemma 2 easily reduces also to a more general assertion:

Corollary 1'. *If X is a perfectly disconnected space, $\text{ind } Y = n$, $f : X \rightarrow Y$ is a perfect mapping, then $\text{ind } X \leq n$.*

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* A space is called perfectly disconnected if each of its points can be represented as the intersection of open-and-closed sets.

Note: Figure translations are in progress. See original paper for figures.

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