

REMARKS ON ONE APPROXIMATE METHOD FOR SOLVING BOUNDARY-VALUE PROBLEMS

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Abstract

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MATHEMATICS

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REMARKS ON ONE APPROXIMATE METHOD FOR SOLVING BOUNDARY-VALUE PROBLEMS

(Presented by Academician I. N. Vekua on 18 IV 1966)

1. In papers ⁽¹⁻³⁾ a new approximate method was developed for solving boundary-value problems

$$Lu = 0 \quad \text{in } G, \quad (1)$$

$$Pu|_S = \psi \quad (2)$$

(where G is a domain with boundary S ; L and P are differential operators, ψ is given, u is the unknown function), which is based on certain integral relations (Green's formulas, Betti formulas, etc.) for solving problem (1)–(2)*:

$$u(x) = \int_S K(x, y) \varphi(y) dS_y + \int_S K_1(x, y) \psi(y) dS_y, \quad u \in S, \quad (3)$$

where the unknown function $\varphi(y)$ satisfies the equation

$$\int_S K(x, y) \varphi(y) dS_y = - \int_S K_1(x, y) \psi(y) dS_y, \quad x \in G_e; \quad (4)$$

G_e is the complement of the domain G to the whole space.

The essential point in justifying the approximate method is the proof of the linear independence and completeness on S of the system

$$\{K(x_i, y)\}, \quad (5)$$

where the points x_i are situated everywhere densely on a certain auxiliary surface S_1 , with respect to which in ⁽¹⁻⁴⁾ it is assumed that it does not touch the surface S . Following ⁽⁵⁾, we shall call the system (5) a **potential system**.

In the present note we shall make several remarks concerning the systems (5).

2. The first remark concerns the question of the stability ⁽⁶⁾ of the systems (5).

Theorem 1. *Every potential system is unstable, i.e., for any $\varepsilon > 0$ there exists an N such that for $n > N$ the Gram determinant G_n of the system (5) will be less than ε .*

In particular, the systems

$$\{\ln r(x_i, y)\}, \quad \left\{ \frac{\partial}{\partial n_y} \ln r(x_i, y) \right\}, \quad \{r^{-1}(x_i, y)\}, \quad \left\{ \frac{\partial}{\partial n_y} r^{-1}(x_i, y) \right\}$$

are unstable. Numerical computations**, carried out for the first of these systems, showed that this property of the system (5) can substantially increase the orthonormalization error, and in some cases lead to emergency stops of the machine because of division by zero.

* We note that in ⁽⁴⁾ the method is applied to the solution of a problem different from (1)–(2).

** Proofs of the results of the first two remarks and the corresponding numerical computations are published in ⁽⁴⁾.

3. The orthonormalized functions corresponding to system (5) can be computed either from the formula

$$\varphi_k = \begin{vmatrix} (\omega_1, \omega_1) & \cdots & (\omega_1, \omega_{k-1}) & \omega_1 \\ \cdots & \cdots & \cdots & \cdots \\ (\omega_k, \omega_1) & \cdots & (\omega_k, \omega_{k-1}) & \omega_k \end{vmatrix} / \sqrt{G_{k-1} G_k}, \quad (6)$$

where $\omega_i = K(x_i, y)$, (ω_i, ω_k) is the scalar product of the functions ω_i and ω_k , or, having constructed the orthonormalized functions $\varphi_1, \varphi_2, \dots, \varphi_{k-1}$, the next function can be found from the formula

$$\varphi_k = \bar{\varphi}_k / \sqrt{(\bar{\varphi}_k, \bar{\varphi}_k)}, \quad (7)$$

where

$$\bar{\varphi}_k = \omega_k + \sum_{i=1}^{k-1} (\omega_k, \varphi_i) \varphi_i.$$

Assuming the functions ω_i to be normalized, one can show that the denominator of formula (6) is equal to

$$\sin \alpha_k \prod_{i=1}^{k-1} \sin^2 \alpha_i,$$

where α_i is the angle between the vector and the hyperplane passing through the vectors $\omega_1, \omega_2, \dots, \omega_{k-1}$, whereas, when formula (7) is applied in orthonormalizing the functions, one has to divide by the number

$$\prod_{i=1}^k \sqrt{(\bar{\varphi}_i, \bar{\varphi}_i)} = \prod_{i=1}^k \sin \alpha_i.$$

It follows directly from this that formula (7) gives, for unreliable systems, a significantly more stable computational scheme than formula (6). Numerical experiments fully confirmed this result.

4. In (6) the phenomenon of “instability” of systems of functions was investigated for the first time, and a class of systems, called “reliable,” was indicated, for which the Ritz method remains stable. Subsequently S. G. Mikhlin ⁽⁷⁾ indicated a considerably broader class (strongly minimal systems) for which the Ritz method retains stability. A system of functions $\{\omega_i\}$ is strongly minimal in H_A ⁽⁸⁾ if the smallest eigenvalue of the Ritz matrix of order n is bounded below by a positive constant independent of n .

Let S and S_1 be concentric circles, and let the points x_i be distributed uniformly on S_1 .

Theorem 2. *The systems $\{\ln r(x_i, y)\}$ and $\left\{ \frac{\partial}{\partial n_y} \right\} \ln r(x_i, y)$ are not strongly minimal in H_E , where E is the identity operator.*

Since for H_E (H_E coincides with L_2) the Ritz matrix coincides with the Gram determinant, Theorem 2 asserts that for any $\varepsilon > 0$ there exists an N such that the smallest eigenvalue in absolute value of the Gram determinant of order n , for $n > N$, will be less than ε .

5. Numerical experiments showed that, by bringing the auxiliary surface S_1 closer to S , we increase the Gram determinant. Thus, for example, if S and S_1 are concentric circles of radii $r = 1$ and $r = 2$, respectively, then we did not succeed in choosing such 10 points x_i that the corresponding Gram determinant G_{10} of order 10 for the system $\{\ln r(x_i, y)\}$ would differ from machine zero (the computations were performed on a BESM-2 machine). When the radius r_1 of the circle S_1 was equal to 1.1, then for G_{24} a value different from machine zero was obtained (although rather small, (10^{-7})). For $r_1 = 1.05$ we obtained $G_{24} \approx 10^{-2}$. Similar results were obtained for

concentric ellipses S and S_1 . Therefore the following question is of interest.
Do discontinuous potential functions form ...

tions (when the points x_i are taken on a sufficiently smooth contour S) form a complete system?

Theorem 3. The system of functions $\{\ln r(x_i, y)\}$, where the points x_i are everywhere dense on S , is linearly independent and complete in the space $L_2(S)$.

Theorem 4. The system of functions $\{\omega_i(y)\}$, where ω_0 is a nonzero constant,

$$\omega_i = \frac{\partial}{\partial n_y} \ln r(x_i, y),$$

and the x_i are everywhere dense on S , is linearly independent and complete in $L_2(S)$.

In the proof one uses the proposition that, for sufficiently smooth contours, the operators

$$\int_S \varphi(y) \ln r(x, y) dS_y, \quad \int_S \varphi(y) \frac{\partial}{\partial n_y} \ln r(x, y) dS_y$$

map the space $L_2(\varphi \in L_2)$ into the Lipschitz space $\text{Lip } \mu$ for $\mu < 1/2$.

Let us consider the system of spatial discontinuous potential functions

$$\{1/r(x_i, y)\}, \quad (8)$$

where the x_i are everywhere dense on a sufficiently smooth surface S . The following theorem holds.

Theorem 5. The system (8) is closed in $L_p(S)$ for $p = 2 - \alpha$ with any $\alpha > 0$ and, consequently, is complete in $L_{p'}(S)$, $p' = (2 - \alpha)/(1 - \alpha)$.

The proof is based on the proposition that the integral operation

$$\iint_S \frac{1}{r(x, y)} \varphi(y) dS_y$$

maps the space $L_{p'}(S)$, $p' = \frac{2 - \alpha}{1 - \alpha}$, for any $\alpha > 0$, into the Lipschitz space $\text{Lip } \mu$ for $\mu < \frac{\alpha}{2 - \alpha}$.

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