

DIFFERENCE INEQUALITIES FOR FUNCTIONS HARMONIC IN A DISK

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

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MATHEMATICS

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DIFFERENCE INEQUALITIES FOR FUNCTIONS HARMONIC IN A DISK

(Presented by Academician S. N. Bernstein on 6 VI 1966)

For definiteness, here the unit disk $Q(0 \leq r \leq 1, 0 \leq d < 2\pi)$ is considered. The well-known theorem of Schwarz ⁽¹⁾ states that, among all functions $U(r, \alpha)$ harmonic inside Q , bounded there in absolute value by one and vanishing at zero at the center of the disk, the function whose boundary values are $U(1, \alpha) = \varphi_0(\alpha - t)$ assumes the greatest value in absolute value at a given point $(r, t) \in Q$, where

$$\varphi_0(\alpha) = \frac{4}{\pi} \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{\cos(2\nu+1)\alpha}{2\nu+1} = \begin{cases} 1, & -\pi/2 < \alpha < \pi/2, \\ -1, & \pi/2 < \alpha < 3/2\pi. \end{cases}$$

We shall use the notation $U_f(r, \alpha)$ for the solution of the Dirichlet problem corresponding to the function $f(t)$ prescribed on the boundary of the disk Q . Then this assertion means that, under the conditions

$$|f(t)| \leq 1, \tag{1}$$

$$\int_0^{2\pi} f(t) dt = 0 \tag{2}$$

the inequality

$$|U_f(r, t) - U_f(0, t)| \leq |U_{\varphi_0}(r, 0) - U_{\varphi_0}(0, 0)| \tag{3}$$

is valid.

Fig. 1

If one considers the difference of the values of $U_f(r, t)$ at two arbitrary points (r_1, t) and (r_2, t) on some radius, then the following proposition holds, generalizing inequality (3):

Theorem 1. Let $0 \leq r_1, r_2 < 1$. In order that, for $r_1 \neq r_2$, the inequality

$$|U_f(r_1, t) - U_f(r_2, t)| \leq |U_{\varphi_0}(r_1, 0) - U_{\varphi_0}(r_2, 0)|$$

hold for every function $f(\alpha)$ satisfying conditions (1) and (2), it is necessary and sufficient that the point with coordinates (r_1, r_2) in the plane $r_1 r_2$ lie on the curve belonging to the square $0 < r_1, r_2 < 1$ with equation

$$r_1^2 r_2^2 - r_2^2 r_1 - r_1^2 r_2 - 2r_1 r_2 - r_2 - r_1 + 1 = 0$$

or below (to the left of) this part of the curve (in the shaded region in Fig. 1).

Let p be an arbitrary natural number; $W^{(p)}$ is the class of functions $f(t)$ of period 2π having an absolutely continuous derivative of order $p - 1$ and such that $|f^{(p)}(t)| \leq 1$ almost everywhere. For functions $U_f(r, \alpha)$ harmonic inside Q and corresponding to functions $f(t) \in W^{(p)}$, there holds

Theorem 2. Whatever the function $f(t) \in W^{(p)}$, for any values r_1, r_2 ($0 \leq r_1, r_2 \leq 1$) the inequality

$$|U_f(r_1, t) - U_f(r_2, t)| \leq \begin{cases} |U_{\varphi_p}(r_1, 0) - U_{\varphi_p}(r_2, 0)|, & p \text{ even,} \\ |U_{\varphi_p}(r_1, \pi/2) - U_{\varphi_p}(r_2, \pi/2)|, & p \text{ odd,} \end{cases}$$

holds, where $\varphi_p(t)$ is that function from $W^{(p)}$ for which $\varphi_p^{(p)} = \varphi_0(t)$.

The indicated extremal role of the functions $\varphi_p(t)$ in the class $W^{(p)}$ is preserved also for differences along arcs concentric with Q of circles.

For an arbitrary natural value k , denote by

$$\Delta_h^k U_f(r, t) = \sum_{\nu=0}^k (-1)^\nu \binom{k}{\nu} U_f(r, t + \nu h)$$

the difference of the function $U_f(r, t)$ with respect to t of order k with step h .

Theorem 3. Whatever the function $f(t) \in W^{(p)}$ and the natural number $k \leq p + 1$, for any positive $h \leq \pi$ everywhere in the disk Q the inequality

$$|\Delta_h^k U_f(r, t)| \leq \begin{cases} |\Delta_h^k U_{\varphi_p}(r, -k(\pi + h/2))|, & p - k \text{ even}, \\ |\Delta_h^k U_{\varphi_p}(r, -k(\pi + h/2) + \pi/2)|, & p - k \text{ odd}. \end{cases} \quad (4)$$

Inequality (4) for any $0 \leq r_1, r_2 < 1$ remains valid also when $p = 0$, $k = 1$ for every function $f(t)$ satisfying condition (1).

Difference inequalities with respect to the radius and with respect to circles analogous to those given above also hold for the harmonic functions $\widetilde{U}_f(r, t)$ conjugate to the functions $U_f(r, t)$, i.e., related to them by the Cauchy-Riemann equations.

Theorem 4. Whatever the function $f(t) \in W^{(p)}$, for any values r_1, r_2 ($0 \leq r_1, r_2 \leq 1$) the inequality

$$|\widetilde{U}_f(r_1, t) - \widetilde{U}_f(r_2, t)| \leq \begin{cases} |\widetilde{U}_{\varphi_p}(r_1, \pi/2) - \widetilde{U}_{\varphi_p}(r_2, \pi/2)|, & p \text{ even}, \\ |\widetilde{U}_{\varphi_p}(r_1, 0) - \widetilde{U}_{\varphi_p}(r_2, 0)|, & p \text{ odd}. \end{cases} \quad (5)$$

Inequality (5) for any r_1, r_2 ($0 \leq r_1, r_2 < 1$) remains valid also when $p = 0$ for every function $f(t)$ satisfying condition (1).

Theorem 5. Whatever the function $f(t) \in W^{(p)}$ and the natural number $k \geq p$, for any positive $h \leq \pi$ everywhere in the disk Q the inequality

$$|\Delta_h^k \widetilde{U}_f(r, t)| \leq \begin{cases} |\Delta_h^k \widetilde{U}_{\varphi_0}(r, -k(\pi + h/2))|, & p - k \text{ odd}, \\ |\Delta_h^k \widetilde{U}_{\varphi_p}(r, -k(\pi + h/2) + \pi/2)|, & p - k \text{ even}. \end{cases}$$

Replacing condition (1) by the condition

$$\int_0^{2\pi} |f(t)| dt \leq 1, \quad (6)$$

and the class $W^{(p)}$ by the class $W^{(p)}L$ of all functions $f(t)$ having an absolutely continuous derivative of order $p - 1$ and such that $\int_0^{2\pi} |f^{(p)}(t)| dt \leq 1$, one can obtain sharp inequalities for the mean oscillations of the corresponding solutions of the Dirichlet problem along radii and along circles concentric with Q .

Theorem 6. If the point (r_1, r_2) belongs to the shaded region in Fig. 1, then

$$\sup_{\int_0^{2\pi} f(t) dt=0, \int_0^{2\pi} |f(t)| dt \leq 1} \int_0^{2\pi} |U_f(r_1, t) - U_f(r_2, t)| dt = \frac{4}{\pi} \left| \sum_{\nu=0}^{\infty} (-1)^\nu \frac{r_1^{2\nu+1} - r_2^{2\nu+1}}{2\nu + 1} \right|.$$

For all natural p and arbitrary values r_1, r_2 ($0 \leq r_1, r_2 \leq 1$), the inequality

$$\sup_{f \in W^{(p)}L} \int_0^{2\pi} |U_f(r_1, t) - U_f(r_2, t)| dt = \frac{4}{\pi} \sum_{\nu=0}^{\infty} (-1)^{\nu(p+1)} \left| \frac{r_1^{2\nu+1} - r_2^{2\nu+1}}{(2\nu+1)^{p+1}} \right|$$

holds.

Theorem 7. For all natural k and p ($k \leq p+1$), for arbitrary values r and h ($0 \leq r \leq 1$, $0 < h \leq \pi$), the equality

$$\sup_{f \in W^{(p)}L} \int_0^{2\pi} |\Delta_h^k U_f(r, t)| dt = \frac{4}{\pi} \left| \sum_{\nu=0}^{\infty} (-1)^{\nu(p+k+1)} \frac{\{2 \sin(\nu+1/2)h\}^k}{(2\nu+1)^{p+1}} r^{2\nu+1} \right|$$

holds.

The equality is also valid in the case $p=0$, $k=1$, $0 \leq r < 1$, if the class $W^{(p)}L$ is replaced by the class of all functions satisfying conditions (2) and (6).

Theorem 8. For all natural p and arbitrary values r_1, r_2 ($0 \leq r_1, r_2 \leq 1$), the equality

$$\sup_{f \in W^{(p)}L} \int_0^{2\pi} |\widetilde{U}_f(r_1, t) - \widetilde{U}_f(r_2, t)| dt = \frac{4}{\pi} \left| \sum_{\nu=0}^{\infty} (-1)^{\nu p} \frac{r_1^{2\nu+1} - r_2^{2\nu+1}}{(2\nu+1)^{p+1}} \right|$$

holds.

The last relation remains valid for $p=0$, $0 \leq r_1, r_2 < 1$, if the class $W^{(p)}L$ is replaced by the class of all functions satisfying conditions (2) and (6).

Theorem 9. For all natural p, k ($k \leq p$), for all values r ($0 \leq r \leq 1$) and $0 < h \leq \pi$, the equality

$$\sup_{f \in W^{(p)}L} \int_0^{2\pi} |\Delta_h^k \widetilde{U}_f(r, t)| dt = \frac{4}{\pi} \left| \sum_{\nu=0}^{\infty} (-1)^{\nu(k+p)} \frac{\{2 \sin(\nu+1/2)h\}^k}{(2\nu+1)^{p+1}} r^{2\nu+1} \right|.$$

The inequalities given here for oscillations of harmonic functions contain a number of different results obtained earlier in other works.

Theorem 1, for $r_2 = 0$, gives the above-mentioned Schwarz inequality for the maximum of the modulus of a harmonic function; and Theorem 4, when $p=0$ and $r_2 = 0$, gives the well-known Kőbe inequality ⁽²⁾ for the maximum of the modulus of a function harmonic in Q , whose conjugate is bounded and has boundary values satisfying conditions (1) and (2). Theorem 2, for $r_1 = 0$, $r_2 = 1$,

gives an inequality equivalent to the known inequalities of Bohr ⁽³⁾ (when $p = 1$) and S. N. Bernstein ⁽⁴⁾ (when $p > 1$) for the maximum of the modulus of functions $f(t) \in W^{(p)}$ satisfying condition (2). For $r_2 = 1$, Theorem 2 gives the sharp estimate obtained in ⁽⁷⁾ of the uniform deviation of a harmonic function from its boundary values. Theorem 6 for natural values of p in the case $r_2 = 1$ was obtained earlier in ⁽⁸⁾ (Chapter VI, § 3, p. 155). Theorems 4 and 8, for $r_1 = 0$, $r_2 = 1$, give inequalities of N. I. Akhiezer and M. G. Krein ⁽⁵⁾ (see also ⁽⁶⁾) for the maximum of the modulus and the mean value of the modulus of functions conjugate to functions $f(t) \in W^{(p)}$.

A special case of Theorem 3, when $r = 1$, $k = 1$, gives an inequality for the modulus of continuity of functions $f(t) \in W^{(p)}$ equivalent to the result obtained in (9) (Theorem 1). For arbitrary k , in the case when $r = 1$, Theorems 3, 5, 7, and 9 give the sharp inequalities obtained in (10) for the moduli of smoothness of higher orders of functions $f(t) \in W^{(p)}$ and of their conjugates.

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