

# ON A MATHEMATICAL MODEL OF PRODUCTION

A. M. Rubinov

1967

SovietRxiv

---

View the original and related papers at <https://sovietsrxiv.org/items/ru-196701.79741>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

## Abstract

## Full Text

UDC 51:330:115

## MATHEMATICS

A. M. RUBINOV

# ON A MATHEMATICAL MODEL OF PRODUCTION

(Presented by Academician L. V. Kantorovich on 21 VII 1966)

**1°.** Many mathematical models of production can be described in terms of mappings that assign to a point of some finite-dimensional space a bounded subset of, generally speaking, another finite-dimensional space.

In this note a general model of this type is investigated. The properties possessed by this model are studied under the assumption that the mappings describing it satisfy certain restrictions. In particular, the question of the asymptotic behavior of optimal trajectories is considered.

For an exact description of the model we introduce some notation. Let  $n_1 \leq n_2 \leq \dots \leq n_t \leq \dots$  be a sequence of natural numbers. Consider a system of finite-dimensional normed spaces

$m_1 \subset m_2 \subset \dots \subset m_t \subset \dots$ . Here  $m_t$  is an  $n_t$ -dimensional space; if  $x = (x^1, \dots, x^{n_t}) \in m_t$ , then  $\|x\| = \max_{i \leq n_t} |x^i|$ . We shall regard the space  $m_t$  as a subspace of  $m_{t'}$  ( $t' > t$ ), spanned by the first  $n_t$  unit vectors. By  $P_{t,t'}$  we denote the projection from  $m_{t'}$  into  $m_t$ . We shall assume that the spaces  $m_t$  are partially ordered in the natural way. By  $K_t$  we denote the cone of positive elements of the space  $m_t$ , and by  $\Xi_t$  the totality of all bounded subsets of  $K_t$ .

The model under consideration is specified by: a) a sequence of natural numbers  $n_1 \leq n_2 \leq \dots \leq n_t \leq \dots$ ; b) a system of mappings  $a_t$  acting from  $K_t$  into  $\Xi_{t+1}$ .

Economically,  $t$  may be interpreted as the number of a time period, and the numbers  $1, 2, \dots, n_t$  as the numbers of products available at the beginning of period  $t$ . Applying the mapping  $a_t$  to the vector  $x = (x^1, \dots, x^{n_t}) \in K_t$  means that the products with numbers  $1, 2, \dots, n_t$ , expended at the beginning of period  $t$  in amounts  $x_1, x_2, \dots, x^{n_t}$ , respectively, can be processed by the end of this period into products with numbers  $1, 2, \dots, n_{t+1}$ ; the set  $a_t(x)$  consists of all output vectors  $y = (y^1, y^2, \dots, y^{n_{t+1}})$  that can be obtained with expenditures  $x$  in period  $t$ . The mappings  $a_t$  may also have a somewhat different meaning, namely, if some part  $z_t$  of the produced product must be used in period  $t$  for consumption, then the set  $a_t(x)$  consists of all vectors  $y - z_t$ , where  $y$  is an output vector that can be obtained with expenditures  $x$  in period  $t$  ( $y \geq z_t$ ).

**2°.** In investigating the model described, meaningful results can be obtained under the assumption that certain restrictions are imposed on the mappings  $a_t$ . Let us list the main such restrictions.

- 1) **Monotonicity:** if  $x', x'' \in K_t$ ,  $x' \geq x''$ , then  $a_t(x') \supset a_t(x'')$ .
- 2) **Upper semicontinuity:** if  $x_n \in K_t$ ,  $x_n \rightarrow x$ ,  $y_n \in a_t(x_n)$ ,  $y_n \rightarrow y$ , then  $y \in a_t(x)$ .
- 3) **Concavity:** if  $x', x'' \in K_t$ ,  $\alpha \in [0, 1]$ , then  $a_t(\alpha x' + (1 - \alpha)x'') \supset \alpha a_t(x') + (1 - \alpha)a_t(x'')$ .
- 4) **Positive homogeneity:** if  $x \in K_t$ ,  $\lambda \geq 0$ , then  $a_t(\lambda x) = \lambda a_t(x)$ .
- 5) The mappings  $a_t$  are bounded in the aggregate: for any positive-

for every  $c$  there is a number  $M_c$ , independent of  $t$ , such that if  $x \in K_t$  ( $t = 1, 2, \dots$ ),  $\|x\| \leq c$ , then  $\sup_{y \in a_t(x)} \|y\| \leq M_c$ .

- b) The sequence  $a_t$  is monotone: if  $t' > t$ , then for  $x \in m_{t'}$ ,

$$a_{t'}(x) \supset a_t(P_{t,t'}x).$$

Let  $\xi \in \Xi_\tau$ . Consider the sets  $A_\tau^t(\xi)$  ( $\tau = 1, 2, \dots$ ;  $t = 0, 1, 2, \dots$ ), defined as follows:  $A_\tau^0(\xi) = \xi$ ;

$$A_\tau^t(\xi) = \bigcup_{x \in A_\tau^{t-1}(\xi)} a_{\tau+t-1}(x) \quad (t = 1, 2, \dots).$$

**Theorem 1.** From the monotonicity of the mappings  $a_t$  follows the boundedness of  $A_\tau^t(\xi)$ ; from the upper semicontinuity of  $a_t$  and the closedness of  $\xi$  follows the closedness of  $A_\tau^t(\xi)$ ; from the concavity of  $a_t$  and the convexity of  $\xi$  follows the convexity of  $A_\tau^t(\xi)$ .

**Consequence.** Since  $A_\tau^1(\{x\}) = a_\tau(x)$ , the assertion of Theorem 1 is also valid for the sets  $a_\tau(x)$ .

In what follows, instead of  $A_1^t(\xi)$  we shall write  $A^t(\xi)$ ; instead of  $A^t(\{x\})$  we shall write  $A^t(x)$ .

Let us note that the scheme described includes, in particular, the Gale model <sup>(1)</sup> and a model of the Gale type <sup>(2)</sup>. Let us also note the closeness of the model considered to the model described by Yu. N. Tyurin <sup>(3)</sup>.

3°. Put  $K_t^* = \{f \in m_t^* \mid f \geq 0\}$ . A functional  $f \in K_t^*$  will be called *prices in period t*. If  $f \in K_t^*$ ,  $\xi \in \Xi_t$ , then put  $f(\xi) = \sup_{x \in \xi} f(x)$ . We shall assume that the mappings  $a_t$  are monotone and upper semicontinuous.

Let  $f_T \in K_T^*$ . By an  $(f_T, x)$ -optimal trajectory we shall mean a finite sequence  $\{x_t\}_{t=1}^T$  such that  $x_1 = x$ ,  $x_t \in a_{t-1}(x_{t-1})$  ( $t = 2, 3, \dots, T$ ). Let  $\xi \in \Xi_1$ ,  $\xi$  be closed. Consider the triple  $(\bar{x}, \bar{\varphi}, \alpha)$ . Here

$$\bar{x} = \{\bar{x}_t\}_{t=1}^\infty \quad (\bar{x}_1 \in \xi, \bar{x}_t \in a_{t-1}(\bar{x}_{t-1}), t = 2, 3, \dots);$$

$$\bar{\varphi} = \{\bar{f}_t\}_{t=1}^{\infty} \quad (\bar{f}_t \in K_t^*, \|\bar{f}_t\| = 1); \quad \alpha = \{\alpha_t\}_{t=1}^{\infty} \quad (0 < \alpha_t < \infty).$$

The triple  $(\bar{\chi}, \bar{\varphi}, \alpha)$  will be called an *equilibrium system* on  $\xi$  if the following conditions are fulfilled: 1)  $\bar{f}_1(\bar{x}_1) > 0$ ; 2)  $\bar{f}_{t+1}(a_t(x)) \leq \alpha_t \bar{f}_t(x)$  ( $x \in A^{t-1}(\xi)$ ); 3)  $\bar{f}_{t+1}(\bar{x}_{t+1}) = \alpha_t \bar{f}_t(\bar{x}_t)$  ( $t = 1, 2, \dots$ ).

Put

$$Z_t(\xi) = \{z = (x, y) \in K_t \times K_{t+1} \mid x \in A^{t-1}(\xi), y \in a_t(x)\};$$

$$W_\varepsilon^t = \{z = (x, y) \in Z_t(\xi) \mid \alpha_t(1 - \varepsilon)\bar{f}_t(x) \geq \bar{f}_{t+1}(y)\}; \quad W_\varepsilon = \bigcup_t W_\varepsilon^t.$$

On the set  $m_t \times m_{t+1}$  introduce a norm by putting, for  $z = (x, y) \in m_t \times m_{t+1}$ ,  $\|z\| = \|x\| + \|y\|$ . The normed space thus obtained will be denoted by  $(m_t \times m_{t+1})_1$ . Consider the linear functional  $\bar{g}_t \in (m_t \times m_{t+1})_1^*$ , defined by the formula  $\bar{g}_t = (\alpha_t \bar{f}_t, -\bar{f}_{t+1})$ , and let  $H_{\bar{g}_t}$  be the hyperplane of the functional  $\bar{g}_t$ .

The introduction of the concept of an equilibrium system is justified by the following theorem.

**Theorem 2.** *Let the mappings  $a_t$  be monotone and upper semicontinuous ( $t = 1, 2, \dots$ ). Consider a closed set  $\xi \in \Xi_1$  such that there exists an equilibrium system  $(\bar{\chi}, \bar{\varphi}, \alpha)$  on  $\xi$ . Let  $x \in \xi$  be such that for some  $\tau$ ,  $\bar{x}_\tau \in A^{\tau-1}(x)$ . Let further  $k_1 \geq k_2$  be arbitrary positive numbers,  $T \geq \tau$ , and let the functional  $f_T \in K_T^*$  have the following properties: a)  $f_T(A^{T-1}(x)) > 0$ ; b)  $k_2 \bar{f}_T \leq f_T \leq k_1 \bar{f}_T$ .*

*Then, for any  $\varepsilon \in (0, 1)$  and any  $(f_T, x)$ -optimal trajectory  $\{x_t\}_{t=1}^T$ , the number of pairs  $(x_t, x_{t+1}) \in W_\varepsilon$  does not exceed the number*

$$N = \ln \left[ \frac{k_2 \bar{f}_1(\bar{x}_1)}{k_1 \bar{f}_1(x)} \right] / \ln(1 - \varepsilon).$$

*If, in addition,*

$$\inf_t \sup_{z \in Z_t(\xi)} \frac{\bar{g}_t(z)}{\|\bar{g}_t\| \|z\|} > 0 \quad \text{and} \quad 0 < \varepsilon < \inf_t \sup_{z \in Z_t(\xi)} \frac{\bar{g}_t(z)}{\|\bar{g}_t\| \|z\|},$$

*the number of pairs  $(x_t, x_{t+1})$  such that*

$$\rho \left( \frac{(x_t, x_{t+1})}{\|(x_t, x_{t+1})\|}, H_{\bar{g}_t} \right) > \varepsilon$$

*also does not exceed the number  $N$ .*

Theorem 2 describes the asymptotic behavior of optimal trajectories. It may be regarded as a generalization of the turnpike theorems in weak form that hold in Gale's model (see, for example, (2)).

4°. Let the mappings  $a_t$  ( $t = 1, 2, \dots$ ) be monotone and upper semicontinuous, and let  $\xi \subset E_1$  be a closed set. An  $(\infty, \xi)$ -optimal trajectory is a sequence

$\{x_t\}_{t=1}^{\infty}$  such that  $x_1 \in \xi$ ,  $x_{t+1} \in a_t(x_t)$ , and  $x_t$  is a maximal element of the set  $A^{t-1}(\xi)$  ( $t = 1, 2, \dots$ ).

**Theorem 3.** Let the mappings  $a_t$  ( $t = 1, 2, \dots$ ) be monotone and upper semi-continuous. Whatever closed set  $\xi \subset E_1$  is given, there exists an  $(\infty, \xi)$ -optimal trajectory.

**Theorem 4.** Let the mappings  $a_t$  ( $t = 1, 2, \dots$ ) be monotone, upper semicontinuous, concave, positively homogeneous, and such that the sets

$$\bigcup_{x \in K_t} a_t(x)$$

are solid. Let the closed convex set  $\xi \subset E_1$  either contain an interior point of the cone  $K_1$ , or consist of a single point and, in addition, possess the property that the sets  $A^t(\xi)$  contain an interior point of the cone  $K_{t+1}$  ( $t = 1, 2, \dots$ ).

Then, whatever the  $(\infty, \xi)$ -optimal trajectory  $\{\bar{x}_t\}_{t=1}^{\infty}$ , there exists an equilibrium system  $(\bar{\chi}, \bar{\varphi}, \alpha)$  on  $\xi$  such that  $\bar{\chi}$  coincides with  $\{\bar{x}_t\}_{t=1}^{\infty}$ .

5°. A sequence  $\{x_t\}_{t=1}^{\infty}$  will be called **consistent** if  $x_t \in K_t$ ,  $\sup_t \|x_t\| < \infty$ , and for any  $t$  and  $t' > t$ ,  $P_{t,t'}x_{t'} = x_t$ .

**Theorem 5.** Let the mappings  $a_t$  ( $t = 1, 2, \dots$ ) be monotone. In order that these mappings be bounded in the aggregate, it is necessary and sufficient that, for every consistent sequence  $\{x_t\}_{t=1}^{\infty}$ ,

$$\sup_t \sup_{y \in a_t(x_t)} \|y\| < \infty.$$

Let  $k \leq n_t$  and  $\varepsilon > 0$ . Put

$$V_{k,t,\varepsilon} = \{x = (x^1, \dots, x^{n_t}) \in m_t \mid \sup_{i \leq k} |x^i| \leq \varepsilon\}.$$

**Theorem 6.** Let the mappings  $a_t$  ( $t = 1, 2, \dots$ ) be monotone and bounded in the aggregate; let the sequence  $a_t$  be monotone. Then for every consistent sequence  $\{x_t\}_{t=1}^{\infty}$ , for every number  $k$  and every  $\varepsilon > 0$ , there is a  $\tau$  such that for  $t \geq \tau$

$$P_{\tau+1,t+1}(a_t(x_t)) \subset a_{\tau}(x_{\tau}) + X_{k,\tau+1,\varepsilon}.$$

The economic meaning of Theorem 6 is as follows: in all time periods  $t \geq \tau$ , the quantity of the  $i$ -th product ( $i = 1, \dots, k$ ) obtained by some method from the vector  $x_t$  will differ from the quantity of this product that can be obtained from the vector  $x_{\tau}$  in period  $\tau$  by no more than  $\varepsilon$ .

The author expresses his gratitude to V. L. Makarov for discussing the present note.

Institute of Mathematics  
Siberian Branch of the Academy of Sciences of the USSR

Received  
9 VII 1966

## REFERENCES

1. D. Gale, Collected volume *Linear Inequalities and Related Questions*, Moscow, 1959.
2. V. L. Makarov, DAN, 165, 767 (1965).
3. Yu. N. Tyurin, *Economics and Mathematical Methods*, 1, 391 (1965).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*