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Abstract**Full Text**

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MATHEMATICS

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QUASI-RANDOM OSCILLATIONS AND THE CAPTURE PROBLEM IN THE RESTRICTED THREE-BODY PROBLEM*(Presented by Academician A. N. Kolmogorov on 20 II 1967)*

1. In paper ⁽¹⁾, J. Chazy classified the final behavior (as $t \rightarrow \infty$) of solutions of the three-body problem of celestial mechanics. Among other logical possibilities, he singled out a class of oscillatory motions, for which at least one of the mutual distances between the bodies is unbounded, although it does not tend to infinity. For a long time the question of the existence of such motions remained open.

In 1954 A. N. Kolmogorov proposed, in order to search for oscillatory motions, carrying out an analysis of the following special case of the three-body problem. Two bodies of equal mass $m_1 = m_2$ are placed symmetrically with respect to OZ and have symmetric velocities. A third body of mass m_3 is situated on the axis OZ itself and its velocity is vertical. From symmetry considerations it follows that m_3 remains on OZ forever, while m_1 and m_2 describe symmetric orbits about it. If $m_3 = 0$, then these orbits are Keplerian.

In 1960 K. A. Sitnikov ⁽²⁾ succeeded in actually proving the existence of oscillations in this model. A. N. Kolmogorov (oral communication) found that Sitnikov's reasoning is based on a very general geometric construction. Developing this point of view further, the author discovered the applicability to this problem of the methods of "symbolic dynamics." Familiarity with Smale's construction ⁽³⁾ proved very useful here, although the assertions of ⁽³⁾ are apparently not directly applicable to the problems considered below.

The present paper is devoted to the investigation of a certain class of differential equations

$$\ddot{x} = -Q(t, x) \tag{1}$$

with right-hand side periodic in t . This class also includes the special case of the three-body problem described above, if $m_3 = 0$. For solutions of equation (1), under the conditions formulated below, the phenomenon of "quasi-randomness" in the sense of ⁽⁴⁾ takes place.

We shall say that in the three-body problem there is the phenomenon of **complete capture** if, as $t \rightarrow +\infty$, the mutual distances between the bodies are bounded, whereas as $t \rightarrow -\infty$ one of the bodies goes off to infinity. J. Chazy and E. Hopf showed that, if complete capture is possible at all, then the corresponding trajectories in phase space form a set of zero Lebesgue measure. For an arbitrary equation (1) an analogous result is contained in Theorem 1. Nevertheless the question of the existence of complete capture remains open. The well-known example of K. A. Sitnikov ⁽⁵⁾ does not relate to this case, since there one of the bodies recedes to infinity both as $t \rightarrow +\infty$ and as $t \rightarrow -\infty$. From the theorems formulated below it follows that complete capture is possible in the restricted problem. It is very likely that the method of proof will also prove applicable in the case of bodies of arbitrary mass.

The class of equations (1) considered below is a natural supplement to the class considered in (4), but differs from it by the existence of infinite (hyperbolic) motions.

2. Consider equations (1), whose right-hand side satisfies the following conditions:

1°. The functions $Q(t, x)$, Q'_x , and Q'_t are continuous for all t and x .

2°. $Q(t, x)$ is odd in x and has period 2π in t .

3°. $Q > 0$ for $x > 0$;

$$\int_0^\infty \int_0^{2\pi} Q(t, x) dt dx < \infty.$$

4°.

$$|Q'_t(t, x)| \leq \psi(x); \quad \int_0^\infty \psi(x) dx < \infty.$$

5°. The ratio $\psi(x)/Q(t, x)^2$ is uniformly bounded for $x \geq x^* > 0$.

6°. $Q'_x \leq 0$ for $x \geq x^* > 0$.

Definition. A solution $x(t)$ is called **hyperbolic** as $t \rightarrow +\infty$ if there exists

$$\dot{x}(+\infty) = \lim_{t \rightarrow +\infty} \dot{x}(t);$$

parabolic if

$$\dot{x}(\infty) = \lim_{t \rightarrow +\infty} \dot{x}(t) = 0, \quad \lim_{t \rightarrow +\infty} x(t) = \infty;$$

oscillatory if $x(t)$ has, for $t > 0$, infinitely many zeros forming a sequence tending to $+\infty$. Analogous definitions apply for $t \rightarrow -\infty$.

From conditions 2° and 3° it follows easily that every nontrivial solution of equation (1), both as $t \rightarrow +\infty$ and as $t \rightarrow -\infty$, belongs to one of the three classes.

Theorem 1. *Almost all (in the sense of Lebesgue measure) solutions that are oscillatory as $t \rightarrow -\infty$ are also oscillatory as $t \rightarrow +\infty$. Almost all solutions that are hyperbolic or parabolic as $t \rightarrow -\infty$ are also such as $t \rightarrow +\infty$.*

Denote by $x(t; v, \tau)$ the solution determined by the initial conditions $x(\tau; v, \tau) = 0$, $\dot{x}(\tau; v, \tau) = v$. For $t > \tau$, this solution either increases monotonically (and then it is hyperbolic or parabolic), or it reaches a certain maximum $X^+(v, \tau)$, and then begins to decrease until it again reaches the value $x = 0$. Put

$$Q_0(x) = \frac{1}{2\pi} \int_0^{2\pi} Q(t, x) dt,$$

$$h^+(v, \tau) = \begin{cases} \frac{\dot{x}(+\infty)^2}{2} + \int_0^\infty Q_0(x) dx, & \text{in the first case,} \\ \int_0^{X^+(v, \tau)} Q_0(x) dx, & \text{in the second.} \end{cases}$$

Analogously one defines the function $h^-(v, \tau)$, associated with the behavior of the solution for $t < \tau$. In the stationary case, when Q does not depend on time, both functions coincide and are the energy integral. In the nonstationary case this is, generally speaking, no longer so.

As in (4), v and τ may be regarded as polar coordinates in a certain auxiliary plane Φ . In this plane the level lines

$$h^\pm(v, \tau) = C > 0$$

are smooth Jordan curves. A special role among them is played by the lines

$$\Pi_0^\pm = \left\{ (v, \tau); h^\pm(v, \tau) = \int_0^\infty Q_0(x) dx \right\}.$$

If the point (v, τ) lies outside (on) the line $\Pi_0^\pm$, then the solution $x(t; v, \tau)$ monotonically and will be hyperbolic (parabolic) as $t \rightarrow +\infty$. If, however, (v, τ) belongs to the domain R_0^+ , bounded by the curve Π_0^+ , then the solution $x(t; v, \tau)$ has at least one zero for $t > \tau$. Let among them τ' be the zero nearest to τ , and let $v' = -\dot{x}(\tau'; v, \tau)$. Then $(v', \tau') \in R_0^-$ —the domain bounded by Π_0^- , and the mapping $S : R_0^+ \rightarrow R_0^-$, which carries (v, τ) into (v', τ') , is a succession function. A fixed point of the mapping S corresponds to a solution $x(t; v, \tau)$ whose antiperiod is equal to some multiple of 2π , and whose period is correspondingly twice as large.

Since S preserves area, the curves Π_0^+ and Π_0^- necessarily intersect.

Theorem 2. *If at the point of intersection (v_0, τ_0) the curves Π_0^+ and Π_0^- have different tangents, then there exists a sequence of points $(v_n, \tau_n) \rightarrow (v_0, \tau_0)$ such that, for $n \geq N$, the solution $x(t; v_n, \tau_n)$ has period $4\pi n$, antiperiod $2\pi n$, and on the interval $(\tau_n, \tau_n + 2\pi n)$ this solution is > 0 .*

3. To arbitrary N and a we assign a graph $G(N, a)$, whose vertices are: a) the points $P_n = (v_n, \tau_n)$ from Theorem 1 for $n \geq N$; b) the points $v^- \in [0, a]$, called **entry points**; c) the points $v^+ \in [0, a]$, called **exit points**.

A **path** in the graph G will mean a sequence of its vertices which may either be infinite in both directions and consist of vertices of type a), or be bounded on the left or on the right or at both ends. In the latter case the beginning must be an entry point, the end an exit point, and the intermediate points must as before be of type a).

Denote the successive zeros of an arbitrary solution $x(t; v, \tau)$ by t_k , numbering them to the left and to the right from $t_0 = \tau$; if the solution is oscillatory as $t \rightarrow +\infty$ ($-\infty$), then the sequence $\{t_k\}$ is unbounded to the right (to the left), while in the opposite case it breaks off at some term.

Theorem 3. *If the curves Π_0^+ and Π_0^- have distinct tangents at the point of intersection, then for every $\varepsilon > 0$ there exist such N and a that to each path $\pi = \{a^{(k)}\}$ in the graph $G(N, a)$ there corresponds a solution $x(t)$ for which:*

- 1) $|t_k - t_{k-1} - 2\pi n_k| < \varepsilon$, if $a^{(k)} = P_{n_k}$;
- 2) $x(t)$ has no zeros for $t > t_{k-1}$ and $\dot{x}(+\infty) = v^+$, if $a^{(k)} = v^+$;
- 3) $x(t)$ has no zeros for $t < t_{k+1}$ and $\dot{x}(-\infty) = v^-$, if $a^{(k)} = v^-$.

By choosing paths in the graph G in different ways, we obtain solutions of different types. In particular, periodic paths correspond here to periodic solutions of equation (1), for example paths $\{a^{(k)}\}$, where all $a^{(k)} = P_n$, correspond to the solution $x(t; v_n, \tau_n)$ itself.

An increase in the interval between two successive returns of the solution to the value $x = 0$ is accompanied by an increase in the amplitude of oscillation, and if $t_{k+1} - t_k \rightarrow +\infty$, then also

$$\sup_{[t_k, t_{k+1}]} |x(t)| \rightarrow +\infty.$$

Therefore, taking a path $\pi = \{P_{n_k}\}$ for which $n_k \rightarrow +\infty$, we obtain an oscillatory solution of equation (1). If, however, the path begins at an entry point and then passes through vertices P_{n_k} with bounded indices, then we obtain a solution which as $t \rightarrow -\infty$ grows without bound in modulus, while as $t \rightarrow +\infty$ remains bounded; this is an analogue of total capture for equation (1).

Since to any sequence of integers $\{n_k\}$ there corresponds a path $\{P_{n_k}\}$ and a solution for which the intervals between successive zeros differ little from $2\pi n_k$, knowing the distances between an arbitrarily large number of successive zeros of the solution, we nevertheless cannot predict the distance to the next zero. Thus quasi-randomness in the sense of (4) is present.

4. The motion of a body of mass $m_3 = 0$ in a particular case of the three-body

problem (Fig. 1) is described by the differential equation

$$\ddot{z} = -z/[z^2 + r^2(t)]^{3/2},$$

where $r(t)$ —the radius vector of the elliptic Keplerian motion—is obtained from the equations

$$r(t) = 1/(1 + e \cos \varphi); \quad d\varphi/dt = (1 + e \cos \varphi)^2; \quad \varphi(0) = 0.$$

The verification of conditions 1° – 6° presents no difficulty. It is more difficult to prove that the curves Π_0^+ and Π_0^- are not tangent at the point of intersection. When the eccentricity $e = 0$, these curves turn into the circle $v = \sqrt{2}$. From symmetry considerations it is clear that, for $e > 0$, they intersect on the ray $\tau = 0$. Computing, at the point $\tau = 0$, $v = \sqrt{2}$, the derivatives

$$\left. \frac{\partial}{\partial e} \left(\frac{dv}{d\tau} \right) \right|_{e=0}$$

along the curves Π_0^+ and Π_0^- , one can show that they do not coincide; and therefore the slopes $dv/d\tau$ of the curves Π_0^+ and Π_0^- are different at this point, at least for small e .

Corollary. In the restricted three-body problem, where $m_1 = m_2$ and $m_3 = 0$, there exist motions with complete capture.

It also follows from Theorem 3 that the four possible types of behavior of a solution—hyperbolic, parabolic, oscillatory, and bounded—as $t \rightarrow -\infty$ and as $t \rightarrow +\infty$ are combined in all 16 logical possibilities.

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