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Abstract

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MATHEMATICS

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EXISTENCE OF A CONVEX SURFACE WITH A PRESCRIBED FUNCTION OF THE PRINCIPAL RADII OF CURVATURE

The subject of the present article is the solution of the question of the existence of a closed convex surface satisfying the condition

$$f(R_1, R_2) = \varphi(n), \quad (1)$$

where R_1 and R_2 are the principal radii of curvature, and n is the unit normal vector to the surface. This problem was considered by Christoffel in the case when $f = R_1 + R_2$, and by Minkowski when $f = R_1 R_2$.

1. The formulation of the final result is somewhat complicated. Therefore we shall postpone it to the end of the article and begin with the proof of the existence of a surface satisfying condition (1), subjecting the functions f and φ to the corresponding restrictions in the course of the proof. First of all, we shall assume that the function f is defined for all positive values of R_1, R_2 , is symmetric ($f(R_1, R_2) = f(R_2, R_1)$), and is strictly increasing, i.e.

$$\partial f / \partial R_1 > 0, \quad \partial f / \partial R_2 > 0. \quad (*)$$

As for the function φ , we assume it to be even ($\varphi(n) = \varphi(-n)$). Both functions f and φ are initially assumed to be analytic. At the end this requirement will be weakened to twice differentiability.

2. A convex surface F is determined by condition (1) uniquely up to parallel translation. If $\varphi(n)$ is an even function, as we assume, then the surface F has a center of symmetry ⁽¹⁾.

Let the surface F undergo an infinitesimal centrally symmetric deformation into a surface F_t with support function $H_t = H + tZ$, where H is the support function of F . Then, if the function f for the surface F_t is stationary at $t = 0$, i.e. $df(R_1(t), R_2(t))/dt = 0$, then $Z \equiv 0$. We shall need this result for the

case when the surface F and its deformation are analytic, and in this form it is essentially contained in the work of A. D. Aleksandrov ⁽²⁾.

3. Let $H(n)$ be the support function of the surface F , satisfying equation (1), on the unit sphere ω . If the center of the surface is taken as the origin of coordinates, then $H(n)$ will be an even function ($H(n) = H(-n)$). It satisfies an equation of elliptic type $\Phi(H_{11}, H_{12}, \dots, H, n) = 0$, which is obtained from condition (1). Ellipticity follows from the monotonicity of f with respect to the variables R_1 and R_2 . The corresponding equation in variations has the form

$$\frac{\partial \Phi}{\partial H_{11}} p_{11} + \frac{\partial \Phi}{\partial H_{12}} p_{12} + \dots + \frac{\partial \Phi}{\partial H} = L(p) = 0. \quad (2)$$

As indicated in item 2, the equation $L(p) = 0$ in the class of even functions p ($p(n) = p(-n)$) has no solutions other than the trivial one $p \equiv 0$. The linear operator L induces on the unit sphere a Riemannian metric

$$ds^2 = \frac{\partial \Phi}{\partial H_{11}} du_1^2 + \frac{\partial \Phi}{\partial H_{12}} du_1 du_2 + \frac{\partial \Phi}{\partial H_{22}} du_2^2,$$

where u_1, u_2 are curvilinear coordinates on the sphere. In view of the symmetry of the surface F , and consequently of the evenness of H , the mapping of the sphere ω onto itself by symmetry with respect to its center is isometric in the metric ds^2 .

The problem of reducing equation (2) on the sphere ω to the canonical form

$$\Delta p + A p_{u_1} + B p_{u_2} + C p = 0, \quad (3)$$

where Δ is the second Beltrami differential parameter, is, as is known, connected with the conformal mapping of the manifold with metric ds^2 onto the sphere. In view of the above-mentioned symmetry of the metric of the manifold M , one may assume that the conformal mapping preserves this symmetry.

4. We shall solve the problem of constructing a surface satisfying condition (1) by the method of continuous continuation with respect to a parameter. To this end we include the function $\varphi(n)$ in a continuous family $\varphi_\lambda(n)$, putting

$$\varphi_\lambda(n) = \lambda \varphi + (1 - \lambda) f(1, 1), \quad 0 \leq \lambda \leq 1.$$

The problem is trivially solvable for $\lambda = 0$. The corresponding surface is the sphere of unit radius. Suppose that the problem is solvable for the function φ_{λ_0} . We shall show that it is then solvable for any function φ_λ when λ is sufficiently close to λ_0 .

The equation for the support function $p_\lambda(n)$ of the surface F_λ may be represented in the form

$$L(\bar{p}) + R(\bar{p}) + (\lambda - \lambda_0)(\varphi(n) - f(1, 1)) = 0, \quad (4)$$

where $\bar{p} = p_\lambda - p_{\lambda_0}$, L is the linear elliptic operator (3) corresponding to the surface F_{λ_0} , and R is a quadratic expression with respect to the function $\bar{p}$ and its derivatives up to the second order.

Passing from equation (4) to an integral equation with the aid of an even fundamental solution of the equation $\Delta \bar{p} = 0$ with a logarithmic singularity at two diametrically opposite points of the sphere, we then obtain

$$\bar{p} + \Omega \bar{p} = A, \quad (5)$$

where

$$\Omega(\bar{p}) = \int_{\omega} K(n, n') \bar{p} d\omega, \quad A = \int_{\omega} K_1(n, n') R d\omega.$$

The kernels K and K_1 are even in both variables. As $|n - n'| \rightarrow 0$,

$$K \sim 1/|n - n'|, \quad K_1 \sim \ln |n - n'|.$$

5. To solve equation (5) we shall apply the method of successive approximations. In this connection denote by $C'_{2,\alpha}$ the space of twice differentiable even (in n) functions on the unit sphere whose second derivatives satisfy a Hölder condition with exponent $\alpha > 0$. The operator Ω , acting in this space, is completely continuous. Since the homogeneous equation $\bar{p} + \Omega \bar{p} = 0$ has no solution in $C'_{2,\alpha}$ except the trivial one (Sec. 3), equation (5) is uniquely solvable in $C'_{2,\alpha}$ for any right-hand side A also belonging to $C'_{2,\alpha}$.

We now define the successive approximations $\bar{p}_k$ by means of the recurrent system

$$\bar{p}_k + \Omega \bar{p}_k = A(p_{k-1}).$$

As the initial approximation take $\bar{p}_0(n) \equiv 0$. The process of successive approximations converges when λ is sufficiently close to λ_0 and gives the solution of our problem for such λ . This solution, by virtue of the analyticity and ellipticity of the original equation, will be analytic.

6. We shall now show that, under certain conditions on the functions f and φ , the problem under consideration is solvable for any λ . For this, obviously, it is enough to guarantee the existence of a priori estimates for the function $p_\lambda(n)$ and its derivatives up to the second order. In view of the closedness and con-

convexity of the surface F_λ , this is equivalent to the existence of positive a priori estimates for the principal radii of curvature. In the author's paper ⁽³⁾ the existence of such estimates was proved under the following conditions. Suppose that, when differentiating the function φ along an arc of a great circle on the unit sphere,

$$a \leq \varphi'^2 \leq b, \quad A \leq \varphi'' \leq B.$$

Then, for the existence of positive a priori estimates for the principal radii of curvature of the surface F_λ , it is sufficient that, for any constants α, β, λ such that $a \leq \alpha \leq b$, $A \leq \beta \leq B$, and $0 \leq \lambda \leq 1$, the inequalities

$$\begin{aligned} \lim_{\substack{R_2=R_2(R_1, n) \\ R_1 \rightarrow \infty}} \left\{ (R_2 - R_1) \frac{\partial f}{\partial R_2} + \frac{\partial^2 f}{\partial R_2^2} \frac{\lambda^2 \alpha}{(\partial f / \partial R_2)^2} \right\} &< \beta \lambda, \\ \lim_{\substack{R_1=R_1(R_2, n) \\ R_2 \rightarrow 0}} \left\{ (R_1 - R_2) \frac{\partial f}{\partial R_1} + \frac{\partial^2 f}{\partial R_1^2} \frac{\lambda^2 \alpha}{(\partial f / \partial R_1)^2} \right\} &> \beta \lambda. \end{aligned} \quad (**)$$

After the problem has been solved for the case of analytic functions f and φ , the analyticity condition can be weakened to the requirement of twice differentiability. For this it is enough to approximate the functions f and φ by analytic ones and, after solving the problem, pass to the limit in the solution. As a result, the following theorem is obtained.

Theorem. *Let $f(R_1, R_2)$ be a twice differentiable, symmetric function strictly increasing in both variables. Then, for any even function φ , provided conditions (**) are fulfilled, there exists a closed convex surface satisfying the condition*

$$f(R_1, R_2) = \varphi(n),$$

where R_1, R_2 are the principal radii of curvature of the surface, and n is the unit vector of the outer normal.

In conclusion, we note that an analogous result can also be obtained in a more general case, when the function f also depends on n , being an even function of this variable. The corresponding equation has the form

$$f(R_1, R_2, n) = \varphi(n).$$

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CITED LITERATURE

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2. A. D. Aleksandrov, *DAN*, **22**, No. 3 (1939).
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Note: Figure translations are in progress. See original paper for figures.

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