

EXTENSION OF FUNCTORS FROM THE CATEGORY OF BANACH SPACES TO THE CATEGORY OF $((\alpha))$ -SPACES

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Abstract

Full Text

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MATHEMATICS

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EXTENSION OF FUNCTORS FROM THE CATEGORY OF BANACH SPACES TO THE CATEGORY OF (α) -SPACES

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1°. Let E be a separable locally convex space and A a bounded absolutely convex set (a ball) in E . By E^A we shall denote the linear hull of A , endowed with the norm

$$\|x\|_A = \inf\{\rho > 0 : x \in \rho A\}.$$

The canonical embedding $i^A : E^A \rightarrow E$ is continuous.

If E^A is a Banach space, then A is called a **Banach ball** ⁽¹⁾, and E^A a Banach subspace in E . The space E is called **locally complete** ⁽¹⁾ if every sequence of its elements contained in some E^A and forming there a Cauchy sequence converges in E . A separable locally convex space E is called a **space of type (α)** , or an (α) -space ⁽¹⁾, if it is locally complete and is a (β) -space ⁽⁴⁾, i.e. representable as an inductive limit of Banach spaces.

The class of all (α) -spaces together with all continuous linear mappings of each of them into each forms a category, which we shall denote by $\mathfrak{A}$. The class $\mathfrak{B}$ of all Banach spaces constitutes a full subcategory in $\mathfrak{A}$.

In the present note we prove the possibility of extending any functor in the category $\mathfrak{B}$ (or defined on $\mathfrak{B}$ and taking values in $\mathfrak{A}$) to a functor in the category $\mathfrak{A}$. Such an extension is, generally speaking, non-unique. The existence of two "extreme" extensions is established. As a consequence, in particular, one obtains a certain description of the totality of functors in $\mathfrak{A}$ whose restrictions to $\mathfrak{B}$ coincide.

2°. Spaces of type (α) were first singled out and studied by D. A. Raikov in ⁽¹⁾. We recall some facts and consequences from that work.

Proposition 1. *Let $\{A\}$ be a complete system, i.e. the system of all Banach balls of the (α) -space E . Then E is the limit of the direct spectrum of its Banach subspaces $\{E^A\}$ with respect to the embeddings*

$$i^A : E^A \rightarrow E.$$

Every separable locally convex space can be locally completed ⁽¹⁾. For spaces of countable character, in particular for normed spaces, local completion coincides with ordinary completion.

The topology of any separable locally convex space E can be (β) -strengthened, i.e. can be strengthened so that, in the new topology, E becomes a (β) -space with the same stock of Banach balls.

Local completeness is stable with respect to (β) -strengthening of the topology. A locally completed (α) -space is again an (α) -space.

Let E and F be locally convex spaces. By $L(E, F)$ we shall denote the vector space of all continuous linear mappings of E into F . The space $L(E, F)$ in the topology of simple convergence will be denoted by $L_s(E, F)$, and its (β) -strengthening by $L_i(E, F)$. The topology of the space $L_i(E, F)$ is called inductive ⁽¹⁾. From Theorems 4.4 and 4.8 of ⁽¹⁾ it follows immediately ...

Proposition 2. *If E is a space of type (α) , and F is locally complete, then $L_i(E, F)$ is an (α) -space.*

In the category of separable locally convex spaces an inverse spectrum always has a limit. By the limit of a direct spectrum in this same category one understands the separable space associated with the limit of this same spectrum in the category of all locally convex spaces.

Proposition 3. *The limit of a direct (respectively, inverse) spectrum in the category $\mathfrak{A}$ is the locally completed (respectively, (β) -strengthened) limit of the same spectrum in the category of separable locally convex spaces.*

Let $\{A\}$ and $\{B\}$ be complete systems of Banach disks in (α) -spaces E and F , respectively. The tensor products $E^A \widehat{\otimes} F^B$ ⁽⁴⁾ form a direct spectrum of Banach spaces. The limit of this direct spectrum in the category $\mathfrak{A}$ is called the tensor (α) -product of the spaces E and F ⁽¹⁾ and is denoted by $E \widehat{\otimes} F$. The space $E \widehat{\otimes} F$, obviously, is an (α) -space.

³⁰. In the definition of a (covariant) functor $S : \mathfrak{C} \rightarrow \mathfrak{A}$ ($\mathfrak{C}$ is a subcategory of $\mathfrak{A}$), in addition to the known conditions, we shall assume that the mapping $\varphi \rightarrow S\varphi$ from the space $L_i(E, F)$ into the space $L_i(SE, SF)$, generated by the functor S , is linear and continuous.

Basic functors. Let E be a fixed space of type (α) . The correspondence $F \rightarrow L_i(E, F)$, where $F \in \mathfrak{A}$, defines a functor (Proposition 2) in the category $\mathfrak{A}$, which will be denoted by Ω_E . On morphisms Ω_E is defined in the natural way. The tensor (α) -product defines a functor Σ_E as follows: $\Sigma_{EF} = E \widehat{\otimes} F$.

It was established in ⁽¹⁾ that the category $\mathfrak{A}$ with the functors Σ_E and Ω_E forms a D -category in the sense of A. S. Schwarz ⁽²⁾. Since the functor Σ_E is left adjoint to the functor Ω_E (and Ω_F to Σ_E is right) in the sense of Kan ⁽⁶⁾, it follows that

Proposition 4 ⁽²⁾. *The functor Σ_E (respectively, Ω_E) is permutable with the limits of direct (respectively, inverse) spectra in the category $\mathfrak{A}$.*

4⁰. Consider a functor $T : \mathfrak{B} \rightarrow \mathfrak{B}$ (or $T : \mathfrak{B} \rightarrow \mathfrak{A}$). The collection P_T of all its extensions to a functor in $\mathfrak{A}$ forms a category if, as $\text{Hom}(T^\gamma, T^\delta)$ ($T^\gamma, T^\delta \in P_T$), one takes all those mappings of T^γ into T^δ whose restrictions to $\mathfrak{B}$ coincide with the identity mapping of the functor T into itself.

Main theorem. *Every functor $T : \mathfrak{B} \rightarrow \mathfrak{B}$ (or $T : \mathfrak{B} \rightarrow \mathfrak{A}$) admits an extension to a functor in $\mathfrak{A}$, and in the category P_T of all its extensions there is an initial object T^i and a final object T^π , i.e. for any $T^\gamma \in P_T$ there exists a unique mapping $\sigma^{i,\gamma} : T^i \rightarrow T^\gamma$ and a unique mapping $\sigma_{\gamma,\pi} : T^\gamma \rightarrow T^\pi$.*

The functor T^i is constructed as follows. Let E be an (α) -space and let $\{E^A\}$ be the direct spectrum of its Banach subspaces determined by a complete system of Banach disks $\{A\}$ in E (Proposition 1). By T^{iE} we denote the limit of the corresponding direct spectrum $\{TE^A\}$ in $\mathfrak{A}$ (Proposition 3).

Let E be a separable locally convex space and let $\{U\}$ be a fundamental system of its closed absolutely convex neighborhoods of zero. By E_U we shall mean the quotient space E/N_U (N_U is the largest subspace contained in U), endowed with the usual normed topology. Denote the completion of E_U by $\widehat{E}_U$. The system of Banach spaces $\{\widehat{E}_U\}$ forms an inverse spectrum, whose limit in the category of locally convex spaces coincides with the completion $\widehat{E}$ of the space E ⁽⁵⁾. If E is a space of type (α) , then the space $T^\pi E$ is defined as the limit of the inverse spectrum $\{T\widehat{E}_U\}$ in the category $\mathfrak{A}$ (Proposition 3).

Denote by $F(\mathfrak{A})$ the category of all functors acting in $\mathfrak{A}$. Let $S \in F(\mathfrak{A})$, and let $S|_{\mathfrak{B}}$ be the restriction of S to $\mathfrak{B}$. From the main theorem it follows immediately:

Proposition 5. *The category $P_{S|_{\mathfrak{B}}}$ forms a spectrum ⁽²⁾ in $F(\mathfrak{A})$, whose projective limit is the initial object S^i , and whose inductive limit is the final object S^π .*

5°. Proposition 6. *If the functor $S : \mathfrak{A} \rightarrow \mathfrak{A}$ is permutable with limits of direct spectra in $\mathfrak{A}$, then $S^i = S$.*

Proposition 7. *If the functor $S : \mathfrak{A} \rightarrow \mathfrak{A}$ is permutable with limits of inverse spectra in $\mathfrak{A}$, then S and S^π coincide on complete spaces of the category $\mathfrak{A}$.*

Example 1. Denote by I the identity functor in the category $\mathfrak{A}$. From Proposition 6 it follows that $I^i = I$. If E is a complete (α) -space, then (Proposition 7) $I^\pi E = E$. In the general case, to obtain $I^\pi E$ one must complete E and then (β) -strengthen its topology.

Example 2. From Propositions 5 and 6 it follows that $\Sigma_E^i = \Sigma_E$.

Example 3. For the functor Ω_E^π the equality

$$\Omega_E^\pi(F) = L_i(E, \widehat{F})$$

holds, where $\hat{F}$ is the completion of the (α) -space F . In particular, if F is complete, then (Propositions 5 and 7) $\Omega_E^\pi(F) = L_i(E, F)$. The set of all mappings $\varphi \in L(E, F)$ that transform some neighborhood of zero in E into a bounded set in F coincides with the union of the spaces $L_i(E, F^B)$ ($\{B\}$ is a complete system of Banach disks in F). The limit of the direct spectrum $\{L_i(E, F^B)\}$ in the category $\mathfrak{A}$ coincides with the space $\Omega_E^i(F)$. In particular, if E is a Banach space, then $\Omega_E^i = \Omega_E$.

Example 4. Each space of numerical sequences l_p ($1 \leq p \leq \infty$) defines in the category $\mathfrak{B}$ the corresponding functor $\bar{l}_p^{(3)}$. Let E be a space of type (α) . Consider the space of all sequences (x_n) from the completion $\hat{E}$ for which the series

$$\sum_{n=1}^{\infty} \alpha_n x_n$$

converges absolutely in $\hat{E}$, whatever numerical sequence $(\alpha_n) \in l_q$, $p^{-1} + q^{-1} = 1$. As a set of elements this space coincides with $\bar{l}_p^\pi(E)$. The set of all sequences (x_n) from E for which the series

$$\sum_{n=1}^{\infty} \alpha_n x_n$$

converges absolutely in some fixed Banach subspace of E , whatever numerical sequence $(\alpha_n) \in l_q$, $p^{-1} + q^{-1} = 1$, is naturally turned into a Banach space. The limit of the direct spectrum in the category $\mathfrak{A}$ of Banach spaces obtained in this way coincides with $\bar{l}_p^i(E)$.

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