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1967

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Abstract

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UDC 517.946

MATHEMATICS

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ON THE ASYMPTOTICS OF SOLUTIONS OF THE DIRICHLET PROBLEM NEAR AN IRREGULAR BOUNDARY

(Presented by Academician V. I. Smirnov on 23 XI 1966)

In this work the asymptotic behavior of solutions of the first boundary-value problem for the equation $-\Delta u = f$ is studied. We restrict ourselves to considering the Laplace operator and the simplest singularities on the boundary of the domain. However, the method admits generalization to second-order elliptic equations with variable coefficients, and also makes it possible to consider cases of more complicated boundary configurations.

1°. Let Ω be a finite domain in R_n with boundary Γ ; let the origin $O \in \Gamma$; $\vec{x} = (x_1, \dots, x_{n-1})$; $x = (\vec{x}, x_n)$; $\varphi_x = |x|^{-1}x$; $\cos \theta = |\varphi_x|^{-1}x_n$; $\Sigma_r = \{x : |x| \leq r\}$; σ_r is the boundary of Σ_r ; K_r is the projection of $\sigma_r \cap \Omega$ from the point O onto σ_1 ; $\nu_n = \text{mes}_n \Sigma_1$; $s_n = \text{mes}_{n-1} \sigma_1$. For definiteness we assume that $\Omega \subset \Sigma_1$, and choose $\delta > 0$ such that $\Omega \cap \delta_\delta \neq \emptyset$.

Everywhere below by a solution we shall mean a solution of the equation $-\Delta u = f$ such that $u \in L_2^1(\Omega \setminus \Sigma_r)$ and $u = 0$ on $\Gamma \setminus \Sigma_r$ for all $r > 0$. We shall call a solution u growing if $u \in L_2^1(\Omega)$. Introduce the notation: $J(r) = \|u(r, \cdot)\|_{L_2(K_r)}$; λ_r, Λ_r are the first and second eigenvalues of the Dirichlet problem for the Beltrami operator in K_r , if $r \leq \delta$. For $r > \delta$ put $\lambda_r, \Lambda_r = 0$. Denote by y_1, y_2 the solutions of the equation $y'' + (n-1)y' - r^{-2}\lambda_r y = 0$ such that $y_1 y_2^{-1} \rightarrow 0$ as $r \rightarrow 0$ and $y_2(1) = 0$.

Theorem 1. a) Let $u \in L_2^1(\Omega)$ be a solution of the equation $-\Delta u = f$, where $f \equiv 0$ in Σ_ε . Then, for $r_1 < r_2 < \varepsilon$, the estimate

$$J(r_1) \leq y_1(r_1) y_1^{-1}(r_2) J(r_2)$$

is valid. b) Let u be a growing solution of the equation $\Delta u = 0$. Then

$$J(r_1) \geq y_2(r_1) y_2^{-1}(r_2) J(r_2)$$

for $r_1 < r_2 < 1$.

Theorem 2. There exists no more than one growing solution of the equation $\Delta u = 0$ satisfying the condition

$$\lim_{r \rightarrow 0} J^{-2}(r) r^{2-n} \int_r^1 \exp \left\{ \int_t^1 \frac{\sqrt{4\Lambda_s + (n-2)^2} + \sqrt{4\lambda_s + (n-2)^2}}{2s} ds \right\} \frac{dt}{t} = \infty.$$

Such a solution preserves its sign in Ω .

If λ_r “sufficiently regularly” tends to a limit $\lambda_0 \leq \infty$ as $r \rightarrow 0$, then one can indicate asymptotic formulas for $y_{1,2}$. These formulas, in combination with Theorem 2, make it possible to obtain the following exact conditions for uniqueness of a nontrivial solution of the equation $\Delta u = 0$, satisfying the inequality $J(r) \leq cy_2(r)$: a) $(\Lambda_r - \lambda_r)r^{-1} \in L(0, 1)$ for $\lambda_0 < \infty$; b) $(\sqrt{\Lambda_r} - \sqrt{\lambda_r})r^{-1} \in L(0, 1)$ for $\lambda_0 = \infty$.

From Theorem 1 one can obtain estimates near the point O for the Green function $G(x, s)$ and the harmonic measure $H(x, E)$, $E \subset \Gamma$, $x \in \Omega$ (cf. ⁽¹⁾).

Theorem 3. For any $\alpha > 1$, $\beta < 1$ there exists a constant $k = k(\alpha, \beta)$ such that

$$\begin{aligned} G(x, s) &\leq ky_1(\alpha|x|)y_1^{-1}(\beta|s|)|s|^{2-n} \quad \text{for } \alpha|x| \leq \beta|s|; \\ H(x, \Gamma \setminus \Sigma_\rho) &\leq ky_1(\alpha|x|)y_1^{-1}(\beta\rho) \quad \text{for } \alpha|x| \leq \beta\rho. \end{aligned}$$

The last inequality makes it possible to obtain an estimate for the modulus of continuity of a harmonic function $v \in C(\bar{\Omega})$ at the point O . Let $\gamma(t)$ be a modulus

continuity of the function $v|_\Gamma$ at the point O . Then, for $x \in \Omega$,

$$|v(x) - v(O)| \leq \gamma(\alpha|x|) + ky_1(\alpha|x|) \int_{\alpha|x|}^1 y_1^{-1}(\beta t) d\gamma(t).$$

2°. We shall give asymptotic formulas for an increasing solution of the equation $\Delta u = 0$. We confine ourselves here to the case when Ω , in a neighborhood of the point O , is a body of revolution $\{x : \theta \leq \theta_r\}$, and θ_r tends to the limit θ_0 as $r \rightarrow 0$. Put $\omega_r = \text{mes}_{n-1} K_r$, $\omega_0 = \lim \omega_r$ as $r \rightarrow 0$, and $\chi_r = \omega_0 - \omega_r$. In deriving the asymptotic formulas one has to require that the function χ_r and its derivatives satisfy, as $r \rightarrow 0$, certain integral decay conditions characterizing the remainder term. Since these general conditions are rather cumbersome, we restrict ourselves here to indicating asymptotics for the particular, but typical, case

$$\chi_r = cr^{c_0}(\ln r)^{c_1}(\ln \ln r)^{c_2} \dots (\ln_k r)^{c_k^*}.$$

Let $\theta_0 \in (0, \pi)$. Then, as $r \rightarrow 0$,

$$u = \begin{cases} r^{2-n-m_0} \left[C_{m_r}^{n/2-1}(\cos \theta) + O\left(\int_0^r |\chi_t| \frac{dt}{t}\right) \right], & \text{if } \chi_r r^{-1} \in L(0,1) \text{ and } c_0 \leq 1; \\ r^{2-n-m_0} \exp \left\{ \mu^{-1} \int_r^1 \chi_t \frac{dt}{t} \right\} \left[C_{m_r}^{n/2-1}(\cos \theta) + O\left(|\chi_r| + \int_0^r \chi_t^2 \frac{dt}{t}\right) \right], & \text{if } \chi_r r^{-1} \notin L(0,1), \chi_r^2 r^{-1} \in L(0,1) \end{cases} \quad (1)$$

where $C_m^{n/2-1}$ is the Gegenbauer function, $m = m_r$ satisfies the equation $C_{m_r}^{n/2-1}(\cos \theta_r) = 0$, and

$$\mu = \frac{s_{n-1}(n-2+2m_0)}{(n-2)^2 \sin^2 \theta_0} \left[C_{m_0-1}^{n/2}(\cos \theta_0) \right]^{-2} \int_{\cos \theta_0}^1 (1-t^2)^{(n-3)/2} \left[C_{m_0}^{n/2-1}(t) \right]^2 dt.$$

If $\theta_0 = 0$ and $\theta_r r^{-1} \in L(0,1)$, then, as $r \rightarrow 0$,

$$u = (r\theta_r)^{(2-n)/2} \exp \left\{ \nu \int_r^1 \theta_t^{-1} \frac{dt}{t} \right\} \left[\left(\frac{\theta_r}{\theta} \right)^{(n-3)/2} J_{(n-3)/2} \left(\nu \frac{\theta}{\theta_r} \right) + O\left(\int_0^r \theta_t \frac{dt}{t}\right) \right], \quad (2)$$

where ν is the least positive zero of the Bessel function $J_{(n-3)/2}$.

We pass to the case $\theta_0 = \pi$. If $n = 3$, $r^{-1} \ln^{-1} \chi_r \in L(0,1)$ and $r^{-1} \ln^{-2} \chi_r \in L(0,1)$, then there exists a solution having, as $r \rightarrow 0$, the form

$$u = r^{-1} \exp \left\{ - \int_r^1 \ln^{-1} \chi_t \frac{dt}{t} \right\} \left[1 - \frac{2 \ln \cos \theta/2}{\ln \chi_r} + O\left(|\ln^{-1} \chi_r| + \int_0^r \ln^{-2} \chi_t \frac{dt}{t}\right) \right]. \quad (3)$$

If $n > 3$ and $\chi_r^{2(n-3)/(n-1)} r^{-1} \in L(0,1)$, then, as $r \rightarrow 0$,

$$u = r^{2-n} \exp \left\{ \frac{(n-3)s_{n-1}}{(n-2)s_n} \nu_{n-1}^{(3-n)/(n-1)} \int_r^1 \chi_t^{(n-3)/(n-1)} \frac{dt}{t} \right\} [1 + o(1)]. \quad (4)$$

Under the same condition, formula (4) admits a refinement, which we do not present because of its cumbersomeness.

If $f(x)$ satisfies the condition $F(r)y_1(r)r^{n-1} \in L(0,1)$, where $F(r) = \sup |f(x)|$ on σ_r , then there exists an increasing solution of the equation $-\Delta u = f$, the principal term of whose asymptotics is determined by formulas (1)–(4).

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We note that for the “spherical” cone $\theta \leq \theta_0 \in (0, \pi)$, asymptotic expansions of solutions of general elliptic problems are given in (2).

For unbounded domains one can obtain analogous asymptotic representations of solutions in a neighborhood of an infinitely distant point. As an example, let us consider a “quasi-cylindrical” domain Ω . Let Ω_t be the section of Ω by the hyperplane $x_n = t \geq 0$. We shall assume that the projection of Ω_t onto $x_n = 0$ is obtained from Ω_0 by a similarity transformation with respect to O with coefficient $\alpha(t)$ and shift by the vector $\vec{\beta}(t) = (\beta^{(1)}(t), \dots, \beta^{(n-1)}(t))$. Because of the cumbersomeness of the exact conditions and of the remainder term, in the general case we give the formula only for the case when α and $\beta^{(i)}$ are functions of the form $ct^{c_0}(\ln t)^{c_1} \dots (\ln_k t)^{c_k}$. Namely, if $\alpha'^2(t)\alpha^{-1}(t) \in L(0, \infty)$, $|\vec{\beta}|^2\alpha^{-2} < c < \infty$, then there exists a solution of the equation $\Delta u = 0$, $u = 0$ on Γ , such that as $x \rightarrow \infty$

$$u(x) = \alpha^{(2-n)/2}(x_n) \exp \left\{ \lambda_0^{1/2} \int_0^{x_n} \frac{dt}{\alpha(t)} \left[\Psi \left(\frac{\vec{x} - \vec{\beta}(x_n)}{\alpha(x_n)} \right) + O \left(|\alpha'(x_n)| + \int_{x_n}^{\infty} \frac{\alpha'^2}{\alpha} dt \right) \right] \right\},$$

where λ_0 is the first eigenvalue, and $\Psi(\vec{x})$ is the eigenfunction of the Dirichlet problem for the Laplace operator in Ω_0 . This formula generalizes, to the case $n > 2$, the asymptotic formula for a conformal mapping of an infinite strip, obtained in (3) under less stringent conditions.

Let us return to the case of a bounded domain Ω . It can be shown that increasing solutions of the equation $\Delta u = 0$ admitting asymptotic representations of the type (1)–(4) have the form

$$u(x) = \lim_{s \rightarrow 0} G(x, s) G^{-1}(x_0, s),$$

where $x_0 \in \Omega$, and $s \rightarrow 0$ along a nontangential path*. Hence it follows that

$$\lim_{s \rightarrow 0} \lim_{x \rightarrow 0} G^{-1}(x, s) y_1(|x|) \Psi_{|x|}(\varphi_x) y_2(|s|) \Psi_{|s|}(\varphi_s) = c \in (0, \infty),$$

where $\Psi_{|y|}(\varphi_y)$ is the first eigenfunction of the Dirichlet problem for the Beltrami operator in $K_{|y|}$, and $x, s \rightarrow 0$ along nontangential paths.

3°. Let us note some applications of the results obtained. First consider the question of the deficiency index of the operator Δ . The case $n = 2$ was studied in (4, 5). Here and below we assume that the boundary Γ is sufficiently regular everywhere except at the point O . Let the operator Δ be defined on $\mathcal{D}(\Delta) = L_2^2(\Omega) \cap \dot{L}_2^1(\Omega)$, and let $\bar{\Delta}$ be the closure of Δ , $\widetilde{\Delta}$ the Friedrichs extension

of Δ , and Δ^* the operator adjoint to Δ . From Theorem 1 it follows that $\bar{\Delta} = \widetilde{\Delta}$ if $n > 3$, or if $n = 3$ and $ry_2(r) \in L_2(0, 1)$. Under certain qualitative restrictions on Γ the latter condition is also necessary. If it is violated, the deficiency index is equal to 1. In this case the asymptotic behavior of the solution of the equation $\Delta^*u = 0$ is described by formulas of the type (1)–(4). If $\bar{\Delta} \neq \widetilde{\Delta}$, then $v \in \mathcal{D}(\Delta^*)$ belongs to $\mathcal{D}(\widetilde{\Delta})$ if and only if $v(s)G^{-1}(x_0, s) \rightarrow c < \infty$ as $s \rightarrow 0$ along a nontangential path. Moreover, v belongs to $\mathcal{D}(\bar{\Delta})$ if and only if $c = 0$. If $n = 3$ and Γ near the point O is a cone with regular boundary, then $\Delta = \bar{\Delta} = \widetilde{\Delta}$ for $\lambda_0 > 3/4$, $\Delta = \bar{\Delta} \neq \widetilde{\Delta}$ for $\lambda_0 < 3/4$, and $\Delta = \bar{\Delta}$, $\dim \mathcal{D}(\bar{\Delta}) \pmod{\mathcal{D}(\Delta)} = \infty$ for $\lambda_0 = 3/4$.

Let Ω in a neighborhood of the point O be a body of revolution, and suppose $(\frac{1}{2}s_n - \omega_r)^2 r^{-1} \in L(0, 1)$. Then the condition

$$\lim_{r \rightarrow 0} \int_r^1 (\frac{1}{2}s_n - \omega_t) \frac{dt}{t} < +\infty$$

is necessary and sufficient in order that all harmonic functions,

* This means that $\lim \rho d^{-1} > 0$, where ρ is the distance from s to $\Gamma \cap \sigma_{|s|}$, d is the diameter of $\Omega \cap \sigma_{|s|}$.

having a minimum at the point O , satisfy the inequality $\lim |x|^{-1}(u(x) - u(0)) > 0$ as $x \rightarrow 0$ along a non-tangential path (cf. (6)).

In conclusion let us consider the question of the validity of the estimate $\|D^2 u\|_p \leq c \|\Delta u\|_p$ (cf. (7)) under the assumption that Γ near O is a cone with regular boundary. This estimate holds for all $p > 1$ if $\lambda_0 \geq 2n$. If, however, $\lambda_0 < 2n$, then it is valid for

$$p < 2n(n + 2 - \sqrt{4\lambda_0 + (n - 2)^2})^{-1};$$

the only exception is the case of a regular boundary when $\lambda_0 = n - 1$, but the estimate is valid for all $p > 1$.

Let us also note that the results obtained in 1°, 2° turn out to be useful in proving pointwise estimates for derivatives of solutions, as well as assertions of the type of the Phragmén–Lindelöf principle.

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Received
9 XI 1966

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Note: Figure translations are in progress. See original paper for figures.

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