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Abstract

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MATHEMATICS

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ON AN INVARIANT EQUIPPING OF A SURFACE OF A SPACE WITH AFFINE CONNECTION

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An invariant equipping of a surface of a space with affine connection (with curvature and torsion) was considered earlier only for hypersurfaces ⁽¹⁻³⁾, for two-dimensional surfaces ⁽⁴⁾, and also for surfaces X_m of spaces L_n , whose dimension n is equal to the dimension of the osculating space to X_m of some order $p = 2, 3, 4, \dots$ ⁽⁵⁾; G. F. Laptev solved this problem for $n < m(m+3)/2$. In the present work a method is indicated for internally equipping a surface X_m of spaces L_n of arbitrary dimension.

Let X_m be given in L_n , with connection object $\Gamma_{jk}^i(x^l)$ ($i, j, k, l, \dots = 1, 2, \dots, n$), by the equations

$$x^i = x^i(u^1, u^2, \dots, u^m), \quad (1)$$

where the functions (1) and Γ_{jk}^i are assumed differentiable a sufficient number of times. As in ^(3, 4), put

$$B_\alpha^i \equiv \partial_\alpha x^i, \quad B_{\alpha\beta}^i \equiv \partial_\beta \partial_\alpha x^i \quad \text{and so on;}$$

$$\bar{B}_{\alpha\beta}^i \equiv B_{\alpha\beta}^i + \Gamma_{(jk)}^i B_\alpha^j B_\beta^k, \quad \bar{B}_{\alpha\beta\sigma}^i \equiv \partial_\sigma \bar{B}_{\alpha\beta}^i + \Gamma_{(jk)}^i \bar{B}_{\alpha\beta}^j B_\sigma^k \quad \text{and so on}$$

$$(\alpha, \beta, \sigma, \dots = 1, 2, \dots, m).$$

B_α^i is a tensor both in X_m and in L_n ; $\bar{B}_{\alpha\beta}^i$ is a tensor only in L_n . Let $T^{(1)}\{B_\alpha^i\}$, $T^{(2)}\{\bar{B}_{\alpha\beta}^i, B_\alpha^i\}, \dots, T^{(p-1)}\{\bar{B}_{\alpha_1 \dots \alpha_{p-1}}^i, \dots, B_\alpha^i\}$ be the spaces tangent to X_m of orders $1, 2, \dots, p-1$, where p is such a natural number that $\dim T^{(p-1)} \leq n$, while $\dim T^{(p-1)}$ plus the number of vectors $\bar{B}_{\alpha_1 \dots \alpha_p}^i$ is greater than n . We shall

assume that each $T^{(q)}$ ($q \leq p-1$) has maximal dimension, i.e. that the vectors (relative to L_n) $B_\alpha^i, \dots, \bar{B}_{\alpha_1, \dots, \alpha_{p-1}}^i$ are linearly independent.

Consider the system of linear equations in m_i

$$B_\alpha^i m_i = 0, \dots, \bar{B}_{\alpha_1 \dots \alpha_{p-1}}^i m_i = 0. \quad (2)$$

It is invariant with respect to coordinate transformations both in X_m and in L_n and has $N = n - \dim T^{(p-1)}$ linearly independent solutions m_i^a ($a, b, \dots = 1, 2, \dots, N$). Then

$$\omega_{\alpha_1 \dots \alpha_p}^a \equiv \bar{B}_{\alpha_1 \dots \alpha_p}^i m_i^a \quad (3)$$

forms a system of N linearly independent tensors in X_m .

If one chooses another system of solutions of (2),

$${}^*m_i^b = \lambda_a^b m_i^a \quad (\det |\lambda_a^b| \neq 0),$$

then $\omega_{a_1 \dots a_p}^a$ also undergoes the linear transformation

$${}^*\omega_{a_1 \dots a_p}^b = {}^b_a \lambda \omega_{a_1 \dots a_p}^a, \quad (4)$$

i.e., the index a for $\omega_{a_1 \dots a_p}^a$ has, with respect to the transformations (4), a tensorial character.

Put

$$A^{a_1 \dots a_m} \equiv \omega_{\underline{a_1} \dots \underline{a_p}}^{a_1} \dots \omega_{\underline{a_1}^m \dots \underline{a_p}^m}^{a_m}, \quad (5)$$

where alternation is performed over the indices underlined by identical signs. Obviously, $A^{a_1 \dots a_m}$ ($a_\alpha = 1, 2, \dots, N$) is a tensor with respect to the transformations (4) (a λ -tensor) and a scalar density of weight p in X_m . We shall now distinguish two cases.

I. If p is even, then $A^{a_1 \dots a_m}$ is symmetric. We shall assume that it has nonzero discriminant. Then, as follows from the works (⁷⁻¹⁰), there exists a λ -tensor $\tilde{A}_{a_1 \dots a_m}$ such that

$$\tilde{A}_{aa_2 \dots a_m} A^{ba_2 \dots a_m} = \delta_a^b.$$

In X_m it is a tensor density of weight $-p$.^{*} We now define in X_m a symmetric tensor density of weight $-p$:

$$\mathfrak{H}_{a'_1 \dots a'_p \dots a_1^m \dots a_p^m} \equiv \omega_{(a'_1 \dots a'_p)}^{a_1} \dots \omega_{(a_1^m \dots a_p^m)}^{a_m} \tilde{A}_{a_1 \dots a_m}. \quad (6)$$

Assuming the discriminant of this density to be nonzero and dividing it by a suitable power of its discriminant, we obtain a “pure” tensor

$$H_{a'_1 \dots a'_p \dots a_1^m \dots a_p^m}.$$

In this case, by the method of Lieber–Ermakov–Kramlet (see (7–9)), we can invariantly associate with $H_{a'_1 \dots a'_p \dots a_1^m \dots a_p^m}$ a connection $\tilde{\Gamma}_{\alpha\beta}^\sigma$, which is invariant in L_n .

II. If p is odd, then $A^{a_1 \dots a_m}$ is skew-symmetric. Consider the system of linear homogeneous equations with respect to $C^{\alpha_1 \dots \alpha_p \alpha_{p+1}}$:

$$C^{\alpha_1 \dots \alpha_p \alpha_{p+1}} \omega_{\alpha_1 \dots \alpha_p}^a = 0 \quad (\alpha_1 \dots \alpha_{p+1} = 1, \dots, m). \quad (7)$$

It is easy to calculate that the number k of equations in this system and the number l of unknowns are connected by the condition $k \leq l$. Indeed, l is equal to the number of combinations with repetitions from m taken p at a time, multiplied by m : $l = \tilde{C}_m^p m$; $k = Nm = (n - \dim T^{(p-1)})m \leq$ (the number of vectors $B_{\alpha_1 \dots \alpha_p}^i + \dim T^{(p-1)} - \dim T^{(p-1)})m$, or $k \leq \tilde{C}_m^p m$, i.e. $k \leq l$. Equality is attained only in the case $n = \dim T^{(p)}$, for which the connection in X_m is introduced by the method of Hlavatý (see (5)). Excluding this case, we have $k < l$, and the system (7) has a certain number K of linearly independent solutions

$$C_u^{\alpha_1 \dots \alpha_p \alpha_{p+1}} \quad (v, u = 1, 2, \dots, K).$$

These tensors** are determined up to a linear transformation:

$$*C_u^{\alpha_1 \dots \alpha_p \alpha_{p+1}} = v \Lambda C_v^{\alpha_1 \dots \alpha_p \alpha_{p+1}}. \quad (8)$$

* To form a pseudotensor “reciprocal” to a given pseudotensor, one may use any of its relative invariants; for example, its multidimensional determinant is convenient (see (10)).

** The tensorial character of the quantities C_u^α is established directly.

Again we shall say that $C_u^a \dots$ relative to u is a Λ -tensor. Analogously to (5), introduce the Λ -tensor

$$B_{u_1 \dots u_m} \equiv C_{u_1}^{\hat{\alpha}_1 \dots \hat{\alpha}_{p+1}} \dots C_{u_m}^{\hat{\beta}_1 \dots \hat{\beta}_{p+1}}, \quad (9)$$

where the hats above again indicate alternation. In this case $B_{u_1 \dots u_m}$ will be symmetric. Assuming the discriminant of (9) to be nonzero, we construct for it the reciprocal Λ -tensor $\tilde{B}^{u_1 \dots u_m}$, and then the tensor density in $\dot{X}_m$

$$\mathfrak{h}^{\alpha'_1 \dots \alpha'_p \dots \alpha_1^m \dots \alpha_p^m} \equiv C_{u_1}^{(\alpha'_1 \dots \alpha'_p)} \dots C_{u_m}^{\alpha_1^m \dots \alpha_p^m} \tilde{B}^{u_1 \dots u_m}, \quad (10)$$

which is again sufficient for introducing a connection $\tilde{\Gamma}_{\alpha\beta}^\sigma$ in $\dot{X}_m$.

We note that the differentiability classes of the functions Γ_{jk}^i and (1), used for constructing $\tilde{\Gamma}_{\alpha\beta}^\sigma$, are respectively equal to C^{p-1} and C^{p+1} . For X_m in L_n , where $n < m(m+3)/2$, we obtain C^1 and C^3 .

Introduce the following system of quantities:

$$\tilde{\omega}_a^{\sigma_1 \dots \sigma_p} \equiv \frac{1}{m!} \omega_{a'_1 \dots a'_p}^{a_1} \dots \omega_{\alpha_1^{m-1} \dots \alpha_p^{m-1}}^{a_{m-1}} A_{\alpha_1 \dots \alpha_{m-1} a} \delta_{1 \dots m}^{\alpha'_1 \dots \alpha_1^{m-1} \sigma_1} \dots \delta_{1 \dots m}^{\alpha'_p \dots \alpha_p^{m-1} \sigma_p},$$

where $\delta_{1 \dots m}^{\alpha'_1 \dots \alpha_1^{m-1} \sigma_1}$ (the indices range from 1 to m) is the generalized Kronecker symbol (contraction of a tensor with $\frac{1}{m!} \delta_{1 \dots m}^{\alpha_1 \dots \alpha_m}$ is equivalent to its alternation with respect to the indices $\alpha_1 \dots \alpha_m$). It is easy to see that

$$\tilde{\omega}_a^{\sigma_1 \dots \sigma_p} \omega_{\sigma_1 \dots \sigma_p}^b = \delta_a^b \quad (a, b = 1, \dots, N). \quad (11)$$

We define the invariant equipment by the spaces

$$\mathfrak{P}^{(2)}\{D_\beta B_a^i\}, \quad \mathfrak{P}^{(3)}\{D_\beta D_\gamma B_\delta^i\}, \dots, \quad \mathfrak{P}^{(p-1)}\{D_{\sigma_1} \dots D_{\sigma_{p-2}} B_{\sigma_{p-1}}^i\}$$

(spanned by the vectors indicated in braces) and by the system of vectors

$$n_a^i \equiv \tilde{\omega}^{\sigma_1 \dots \sigma_p} D_{\sigma_1} \dots D_{\sigma_{p-1}} B_{\sigma_p}^i.$$

Here D_σ is the operator of mixed differentiation with respect to the pair of connections Γ_{jk}^i , $\tilde{\Gamma}_{\alpha\beta}^\sigma$, where $\tilde{\Gamma}_{\alpha\beta}^\sigma$ has been introduced by the method described above. It is easy to see that, by virtue of the assumptions made at the beginning of the paper and of (11), we obtain a genuinely invariant equipment. The connection $\Gamma_{\alpha\beta}^\sigma$ induced by this equipment, generally speaking, does not coincide with $\tilde{\Gamma}_{\alpha\beta}^\sigma$.

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