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Abstract

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MATHEMATICS

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INTEGRAL GEOMETRY IN EUCLIDEAN SPACE AND ITS CONNECTION WITH BOUNDARY-VALUE PROBLEMS

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In the present note we consider the real analogue of the problem of reconstructing a function, defined in three-dimensional affine space, from its known integrals over lines belonging to a certain complex (a three-parameter family), solved for the case of complex affine space by I. M. Gelfand and M. I. Graev ^(1, 2). A connection is established between this problem and boundary-value problems for a differential equation.

1. Let R_3 denote real Euclidean space, and let Π_4 be the manifold of its lines. We shall agree to specify each point in Π_4 , i.e. a line in R_3 , by a direction vector $l \neq 0$ and a point a of R_3 lying on this line. Thus we define a mapping

$$p: (R_3 \setminus \{0\}) \times R_3 = A_6 \rightarrow \Pi_4.$$

We shall agree, for brevity, to call any function or differential form on A_6 that coincides with the image of some function or differential form on Π_4 under the mapping p^* a function or differential form defined on Π_4 . Let (x, y) denote the scalar product in R_3 , and let ∂_x be the operator which assigns to a function of x its gradient. It is easy to see that, in order that a function or differential form on A_6 be defined on Π_4 , it is necessary and sufficient that it be invariant with respect to the transformations

$$l \rightarrow tl, \quad a \rightarrow a + sl, \quad (1)$$

where $t \neq 0$ and s are arbitrary real parameters. For differentiable functions this condition is equivalent to the following:

$$(l, \partial_l)h \equiv l_1 \frac{\partial h}{\partial l_1} + l_2 \frac{\partial h}{\partial l_2} + l_3 \frac{\partial h}{\partial l_3} = 0, \quad (l, \partial_a)h \equiv l_1 \frac{\partial h}{\partial a_1} + l_2 \frac{\partial h}{\partial a_2} + l_3 \frac{\partial h}{\partial a_3} = 0. \quad (2)$$

2. The manifold Π_4 is naturally a pseudo-Riemannian space with metric

$$ds^2 = (l, dl, da)/(l, l) \quad (3)$$

(the numerator here is the mixed product in R_3). This metric corresponds to the volume element

$$\omega = \frac{(l_1 dl_2 \wedge dl_3 + l_2 dl_3 \wedge dl_1 + l_3 dl_1 \wedge dl_2) \wedge (l_1 da_2 \wedge da_3 + l_2 da_3 \wedge da_1 + l_3 da_1 \wedge da_2)}{(l, l)^2}, \quad (4)$$

the Laplace-Beltrami operator

$$\square = (l, \partial_l, \partial_a) \equiv l_1 \left(\frac{\partial^2}{\partial l_2 \partial a_3} - \frac{\partial^2}{\partial l_3 \partial a_2} \right) + l_2 \left(\frac{\partial^2}{\partial l_3 \partial a_1} - \frac{\partial^2}{\partial l_1 \partial a_3} \right) + l_3 \left(\frac{\partial^2}{\partial l_1 \partial a_2} - \frac{\partial^2}{\partial l_2 \partial a_1} \right), \quad (5)$$

and the codifferentiation operator δ (divergence), whose explicit form we shall not write down, but only note that if, as usual, we denote $h\omega = *h$, then

$$\delta * h = \frac{(l_1 dl_2 \wedge dl_3 + l_2 dl_3 \wedge dl_1 + l_3 dl_1 \wedge dl_2) \wedge (\partial_a, \partial_l)h}{(l, l)} + \alpha, \quad (6)$$

where a is a certain closed form. In this case, as is not hard to notice, $d\delta * h = *\square h$.

Let us note that on Π_4 , besides $\square$, there also exist other invariant differential operators of second order. We shall encounter, for example, the operator

$$\hat{\Delta} = (\partial_a, \partial_a) = \partial^2 / \partial a_1^2 + \partial^2 / \partial a_2^2 + \partial^2 / \partial a_3^2. \quad (7)$$

We shall call a function $h(l, a)$, defined on Π_4 , rapidly decreasing if it itself and all its products by arbitrary polynomials on Π_4 in the coordinates of the vector a are integrable over the whole space Π_4 . The set of all infinitely differentiable rapidly decreasing functions on Π_4 will be denoted by $\hat{S}$. By $\hat{K}$ we shall denote the set of all infinitely differentiable functions on Π_4 with compact support. In the linear space $\hat{S}$ one introduces, as is customarily done in the theory of generalized functions, the topology of uniform convergence of all derivatives on compact sets, and in $\hat{K}$ the topology of the inductive limit of subspaces in $\hat{S}$ consisting of functions whose supports lie in one and the same compact set (cf., for example, ⁽³⁾). We shall call a function from $\hat{S}$ or $\hat{K}$ harmonic if it is taken to zero by the operator $\square$. The subspaces of harmonic functions in $\hat{S}$ or $\hat{K}$ will be denoted by $H\hat{S}$ or $H\hat{K}$, respectively.

3. Let S and K denote, respectively, the spaces of infinitely differentiable and finite functions on R_3 (see ⁽³⁾). To each function $f(x) \in S$ we assign its integrals over all possible straight lines in R_3 :

$$\hat{f}(l, a) = |l| \int_{-\infty}^{\infty} f(a + lt) dt. \quad (8)$$

Theorem 1. *The operation $f \rightarrow \hat{f}$ is a one-to-one and continuous mapping of the spaces S and K onto $H\hat{S}$ and $H\hat{K}$, respectively. Moreover,*

$$(\Delta f)^{\wedge} = \hat{\Delta} \hat{f} * . \quad (9)$$

Let us note that the essential part of this theorem is due to F. John, who in 1938 established that integrals over straight lines satisfy a system of equations equivalent to (2), (6), and also that every function on Π_4 satisfying this system is representable in the form (8) (see ⁽⁴⁾).

4. If x is an arbitrary point in R_3 , and $p(l, a)$ is a point in Π_4 , i.e. a straight line in R_3 , then the Euclidean distance between them is equal to

$$\rho(x; l, a) = \frac{|[l, a - x]| **}{|l|}. \quad (10)$$

For each function $h(l, a) \in \hat{S}$ put

$$\check{h}(x) = -\frac{1}{4\pi^3} \int_{\Pi_4} \rho^\lambda(x; l, a) h(l, a) \omega|_{\lambda=-3}. \quad (11)$$

The right-hand side here must be understood in the following way. The integral here converges and defines an analytic function of λ for $\text{Re } \lambda > -2$. The right-hand side of (11) is the value at $\lambda = -3$ of the analytic continuation of this integral.

Theorem 2. *The operation $h \rightarrow \check{h}$ is a continuous mapping of the spaces $\hat{S}$ and $\hat{K}$ onto S and K , respectively. The kernel of this mapping is the orthogonal complement in the spaces $\hat{S}$ and $\hat{K}$ of the subspaces $H\hat{S}$ and $H\hat{K}$ in the sense of the scalar product*

$$(h_1, h_2) = \int_{\Pi_4} \bar{h}_1 h_2 \omega. \quad (12)$$

* Δ denotes the Laplace operator in R_3 .

** $[a, b]$ denotes the vector product in R_3 .

In this case

$$(\Delta \hat{h})^\vee = \Delta \check{h}. \quad (13)$$

Theorem 3. The operation $h \rightarrow \hat{h}$ is a one-to-one and continuous mapping of the spaces $H\hat{S}$ and $H\hat{K}$ onto S and K , respectively.

In the case $h \in H\hat{S}$, formula (11) is equivalent to the following:

$$\check{h}(x) = -\frac{1}{2\pi^2} \int_{[m[l, a-x]] = 0} p^\lambda(x; l, a) h(l, a) \theta|_{\lambda=-2}, \quad (14)$$

where m is an arbitrary vector from $R_3 \setminus \{0\}$,

$$\theta = (m, l, dl) \wedge (m, l, da) / (m, m) |l|^3. \quad (15)$$

We note that the integration in (14) is carried out over the set of lines lying in a plane passing through x ; the independence of the integral (14) from the choice of such a plane is a necessary and sufficient condition for the harmonicity of the function $h(l, a)$.

Theorem 4. For every function $f \in S$ the inversion formula

$$f = (\hat{f})^\vee, \quad (16)$$

holds, and for every function h from $\hat{S}$ the mapping $h \rightarrow (\check{h})^\wedge$ is the orthogonal projection onto $H\hat{S}$.

5. We now turn to the problem of reconstructing a function $f \in K$ (or S) from its known integrals over lines belonging to some complex. Formulas (16) and (14) make it possible to solve it for the special case of a complex that is a one-parameter family of plane fields. For example, for the complex of lines intersecting a fixed line, the reconstruction problem is solved completely in this way, while for the complex of lines tangent to a cone, the value of the function $f(x)$ is reconstructed only at points lying outside this cone.

Of course, it is very interesting to find an inversion formula for a broader class of complexes. Instead of this problem we shall further consider the equivalent boundary-value problem for the Laplace equation in Π_4 : knowing the values of the function $h \in H\hat{K}$ on some hypersurface D in Π_4 , find this function (the equivalence of these problems follows directly from Theorems 3 and 4). It turns out, however, that the stated problem is solvable only for a comparatively narrow class of hypersurfaces. In the general case, in order to find a function $h \in H\hat{K}$, one must know not only its values on D , but also the values of its normal derivatives (in the sense of the metric (3)) on this hypersurface.

6. To solve the boundary-value problem just formulated, we shall now propose a certain analogue of Green's formula. Let us begin with some preliminaries.

Let $p(m, b)$ be a fixed point of Π_4 , and let k be some vector from $R_3 \setminus \{0\}$. Consider a current of degree zero (a generalized function) assigning to each compactly supported 4-form φ the number

$$\langle \varphi, T^\lambda(k; m, b; l, a) \rangle = \int_{\Pi_4} \left\{ \frac{(l, m, a - b) \operatorname{sgn}\{(l, k)(m, k)\} + i0}{|l| |m|} \right\}^\lambda \varphi. \quad (17)$$

The integral here converges for $\operatorname{Re} \lambda > 0$; for the remaining λ it must be understood in the regularized sense, i.e. as an analytic continuation in λ . Let C denote some 4-chain, which, as usual (see (5)), we identify with the current

$$\langle \alpha, C \rangle = \int_C \alpha.$$

Represent this chain in the form

$$C = \sum_{i \in I} t_i C_i,$$

where C_i are elements of the chain, and assign to each $i \in I$ a vector $k_i \in R_3 \setminus \{0\}$ such that for all points $p(l, a)$ belonging to the set $|C_i|$ of points of the element C_i , $(l, k_i) \neq 0$. The existence of the vector k_i follows from the fact that the set $|C_i|$ is situated in a single coordinate neighborhood. Put now

$$T_i^\lambda(m, b; l, a) = t_i T^\lambda(k_i; m, b; l, a). \quad (18)$$

We shall say that the point $p(m, b)$ is situated regularly with respect to the chain C if, in a neighborhood of any point of intersection of the isotropic cone with vertex $p(m, b)^*$ and the boundary ∂C_i of an arbitrary element C_i of the chain C , the cone and the boundary ∂C_i can serve as a pair of coordinate neighborhoods in some system of local coordinates. If this condition is fulfilled, then the products of currents

$$\sum_{i \in I} T_i^\lambda \partial C_i, \quad \sum_{i \in I} \delta * T_i^\lambda \wedge \partial C_i \quad (19)$$

are defined.

It is clear that the common support of the currents (19) is the hypersurface D in Π_4 lying on the union of all $|\partial C_i|$; generally speaking, this surface depends on the representation of the chain C and on the choice of the vectors k_i ; it may be larger than $|\partial C|$.

Theorem 5. If the point $p(m, b)$ is situated regularly with respect to the chain $C = \sum_{i \in I} t_i C_i$, and if it lies inside the set $|C|$, then for any function $h \in \hat{H} \hat{K}$

$$h(m, b) = \frac{1}{\pi^2} \sum_{i \in I} \{ \langle h, \delta * T_i^\lambda \wedge \partial C_i \rangle - \langle \delta * h, T_i^\lambda \partial C_i \rangle \}_{\lambda=-1}. \quad (20)$$

Let us note that, in order to compute the first term on the right, it is necessary to know the values of h on D , while in order to compute the second term, the values on D of the normal derivatives of this function are needed. However, if D is given by the equation $F(l, a) = 0$, with $\delta * F = 0$ on D , then these normal derivatives can be found from the values of h on D .

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* That is, the cone consisting of all points at zero distance from $p(m, b)$.

Note: Figure translations are in progress. See original paper for figures.

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