

ON LOCAL COHOMOLOGY GROUPS

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1967

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Abstract

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UDC 513.836

MATHEMATICS

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ON LOCAL COHOMOLOGY GROUPS

(Presented by Academician P. S. Aleksandrov, 19 XII 1966)

Let X be a paracompact Hausdorff space and $x \in X$ a certain fixed point. Denote by I_x^p the local cohomology group at the point x , obtained as the inductive (direct) limit of the cohomology groups $H^p(U \setminus x; G)$ over the directed set of all open neighborhoods U of the point x (G is an arbitrary abelian coefficient group, the cohomology is that of Aleksandrov-Čech); for $p = 0$ in this definition one uses the reduced groups $\widetilde{H}_0(U \setminus x; G)$.

The local groups I_x^p were first considered by Conner ⁽¹⁰⁾, and then by A. Borel ⁽⁷⁾, who, in particular, applied them to obtain certain local fixed-point theorems in the theory of transformation groups. These groups were also studied by Raymond ^(12,13), who used them in the study of Freudenthal compactifications of generalized manifolds, and also for proving various kinds of duality laws in generalized manifolds.

On the other hand, the following definition of local cohomology groups is widely known: the group H_x^p is the projective (inverse) limit of the cohomology groups $H^p(X, X \setminus U; G)$ (over the directed set of all neighborhoods of the point x).

The analogous definition in the case of homology goes back to P. S. Aleksandrov ⁽⁶⁾ and has been studied by many authors (see, for example, ^(2,9)). Implicitly, the definition of the groups H_x^p is contained in Wilder's book ⁽¹⁵⁾, Chapter 10, where a definition of local Betti numbers is given, and also in the theory of cohomology manifolds (see ^(15,8)). Articles ^(11,14) and others are devoted to the study of these groups.

For cohomology manifolds (over an arbitrary field K) A. Borel ⁽⁷⁾ and Raymond ⁽¹³⁾ proved that the groups I_x^p (as well as H_x^{p+1}) are zero for all $p \neq n-1$, while the group I_x^{n-1} is isomorphic to K (n is the dimension of the manifold).

In this note it is shown that there exists a natural homomorphism $\delta_x^p : I_x^p \rightarrow H_x^{p+1}$, and its structure is studied (Theorem 1). In the case when G is a module of finite type over an Artinian ring (i.e., a ring satisfying the minimum condition), conditions are indicated under which the mapping is epimorphic and monomorphic (Corollary 2, Theorem 2).

The connection between the local cohomology groups I_x^p and H_x^{p+1} can be established as follows. For any neighborhood U of the point x , consider the Leray spectral sequence of the inclusion mapping

$$i : (X \setminus x, X \setminus U) \subset (X, X \setminus U)$$

(as coefficient group one may use any abelian group G). The direct images $R^q i_* G$ of the constant sheaf G over $U \setminus x$, for $q > 0$, as well as $R^0 i_* G/G$, have only one nontrivial stalk, over the point x . Therefore, using the arguments given before Theorem 1 in ⁽⁴⁾, and repeating the first part of the proof of Theorem 1, concerning the derivation of the exact sequence $(*)$ (see ⁽⁴⁾), we obtain the following exact sequence of cohomology groups—

cohomologies*:

$$0 \rightarrow H^0(X, X \setminus U; G) \rightarrow H^0(X \setminus x, X \setminus U; G) \rightarrow I_x^0 \rightarrow \quad (*)$$

$$\rightarrow H^1(X, X \setminus U; G) \rightarrow \dots \rightarrow H^p(X, X \setminus U; G) \rightarrow$$

$$\rightarrow H^p(X \setminus x, X \setminus U; G) \rightarrow I_x^p \rightarrow \dots$$

If V is a neighborhood of the point x contained in U , then, as is not difficult to see, there exists a homomorphism of the sequence $(*)$ corresponding to the set V into the sequence corresponding to the neighborhood U ; this homomorphism coincides (in the corresponding terms of the sequence) with the cohomology mapping induced by the inclusion of pairs

$$(X, X \setminus U) \subset (X, X \setminus V), \quad (X \setminus x, X \setminus U) \subset (X \setminus x, X \setminus V).$$

The groups I_x^p are mapped identically, since the sheaves $R^p i_* G$ over $(X, X \setminus U)$ are induced by these same sheaves over $(X, X \setminus V)$ under the inclusion $(X, X \setminus U) \subset (X, X \setminus V)$. The commutativity of the mappings is easy to verify.

Passing to the limit over the directed set of all neighborhoods of the point x in the space X , we obtain a certain homomorphism

$$\delta_x^p : I_x^p \rightarrow H_x^{p+1}.$$

Let us determine the structure of this homomorphism.

Suppose that the point x is the intersection of a countable number of some of its neighborhoods in the space X (this is always the case if the space X is perfectly normal). In this case the following proposition is valid**.

Lemma. *The projective (i.e., inverse) limit of the system of cohomology groups $H^p(X \setminus x, U \setminus x; G)$ is equal to zero for all $p \geq 0$.*

Proof. First of all, note that the group $H^p(X \setminus x, U \setminus x; G)$ is isomorphic to

$$H_{\Phi|U}^p(X \setminus x; G),$$

where Φ is the family of all closed subsets in $X \setminus x$, and $\Phi|U$ is the collection of all such subsets contained in U (notation from ⁽¹⁾). Let $C^*(X \setminus x; G)$ be the canonical resolution of the constant sheaf G over $X \setminus x$. Then, for every open set U containing x , the restriction of the resolution $C^*(X \setminus x; G)$ to $U \setminus x$ is, evidently, the canonical resolution of the constant sheaf G over $U \setminus x$. Therefore, for every pair of neighborhoods $V \subset U$ of the point x , and for any $p \geq 0$, the homomorphism of p -dimensional chains

$$C_{\Phi|V}^p(X \setminus x; G) \rightarrow C_{\Phi|U}^p(X \setminus x; G)$$

is a monomorphism.

Let h^p be an arbitrary element of the limit of the groups

$$H_{\Phi|U}^p(X \setminus x; G),$$

and let h_n^p be its image in

$$H_{\Phi|U_n}^p(X \setminus x; G).$$

Denote by c_n^p an element representing h_n^p (a cocycle) in the group of cochains

$$C_{\Phi|U_n}^p(X \setminus x; G)$$

(here U_n are neighborhoods from a countable system). By assumption, the elements c_n^p and c_{n+1}^p , considered in

$$C_{\Phi|U_n}^p(X \setminus x; G), \quad p \geq 1,$$

are cohomologous to each other:

$$c_n^p - c_{n+1}^p = da_n^{p-1},$$

where a_n^{p-1} is some cochain from

$$C_{\Phi|U_n}^{p-1}(X \setminus x; G),$$

which is a section of the $(p-1)$ -dimensional component of the canonical resolution of the constant sheaf G over $X \setminus x$, whose support lies in U_n . Since at each point $y \in X \setminus x$ only a finite number of such sections is nonzero, for every n the section

$$\beta_n^{p-1} = \sum_{i \geq n} a_i^{p-1}$$

is defined.

It is evident that

$$d\beta_n^{p-1} = c_n^p.$$

This means that, for every open set U_n , the element h_n^p is equal to zero; all the more, the element h^p is equal to zero. The case $p = 0$ is evident.

* As shown in ⁽¹³⁾, the groups I_x^p are isomorphic to the groups $H^{p+1}(X, X \setminus x; G)$, and therefore the sequence (*) can also be obtained as the exact sequence of the triple $(X, X \setminus x, X \setminus U)$.

** The authors have not been able to prove this without the indicated countability restriction.

With the help of this lemma it is easy to establish the structure of the homomorphism δ_x^p . Let J_U^p be the kernel of the homomorphism

$$\delta_U^p : I_x^p \rightarrow H^{p+1}(X, X \setminus U; G)$$

and let H_U^{p+1} be the kernel of

$$H^{p+1}(X, X \setminus U; G) \rightarrow H^{p+1}(X \setminus x, X \setminus U; G).$$

Theorem 1. The group H_x^{p+1} is equal to the projective limit of the system of groups H_U^{p+1} ; moreover, for every pair of neighborhoods $V \subset U$ of the point x in X the map

$$H_V^{p+1} \rightarrow H_U^{p+1},$$

induced by the inclusion $(X, X \setminus U) \subset (X, X \setminus V)$, is an epimorphism. The homomorphism

$$\delta_x^p : I_x^p \rightarrow H_x^{p+1}$$

coincides with the limiting map of the system of homomorphisms

$$\delta_U^p : I_x^p \rightarrow I_x^p / J_U^p = H_U^{p+1}$$

(in particular, the kernel of δ_x^p is equal to $\bigcap_U J_U^p$).

Proof. The first assertion of the theorem follows from the lemma and from the fact that, in passing to the projective limit in a projective system of homomorphisms, the kernel of the limiting homomorphism is equal to the limit of the kernels of the homomorphisms of the system (see, for example, (5), § 5, Ch. 8). The second assertion follows from the fact that, for any pair of neighborhoods $V \subset U$ of the point x , there is an inclusion $J_V^p \subset J_U^p$, and the homomorphism

$$H_V^{p+1} \rightarrow H_U^{p+1}$$

coincides, by exactness of the sequence (*), with the mapping of factor groups

$$I_x^p / J_V^p \rightarrow I_x^p / J_U^p.$$

The remaining assertions are obvious.

Corollary 1. If $I_x^p = 0$, then also $H_x^{p+1} = 0$.

If, as the coefficient group G , one takes some module of finite type over a ring with the minimality condition (2), for example a field or an arbitrary finite group, then the following will also be true.

Corollary 2. If the numbers p_x^{p+1} do not exceed some fixed N , then the map

$$\delta_x^p : I_x^p \rightarrow H_x^{p+1}$$

is an epimorphism; moreover, there is a neighborhood U of the point x for which the group H_x^{p+1} coincides with the kernel of the homomorphism

$$H^{p+1}(X, X \setminus U; G) \rightarrow H^{p+1}(X \setminus x, X \setminus U; G).$$

For the definition of the inequalities $p_x^k \leq N$ and $p_x^k < \infty$, see (7).

We now clarify the conditions under which the map δ_x^p has no kernel.

Theorem 2. If $p_x^p < \infty$, and G is a module of finite type over a ring with the minimality condition, for example any finite group or field, then the mapping

$$\delta^p : I_x^p \rightarrow H_x^{p+1}$$

is a monomorphism.

Proof. In view of the lemma, it is enough to show that, under the hypotheses of the theorem, exactness in the term I_x^p of the sequence (*) is preserved under passage to the projective limit. Let j be any element of the kernel of the homomorphism δ_x^p , and let J be the submodule in I_x^p generated by this element. Obviously, J is contained in the kernel of the homomorphism

$$\delta_U^p : I_x^p \rightarrow H^{p+1}(X, X \setminus U; G)$$

for every neighborhood U of the point x . Since the sequence (*) is exact, J is contained in the image of the homomorphism

$$\tilde{H}^p(X \setminus x, X \setminus U; G) \rightarrow I_x^p.$$

Let A_U be the submodule of the module $H^p(X \setminus x, X \setminus U; G)$ which is the full inverse image of J , and let B_U be the kernel of the homomorphism $A_U \rightarrow J$. Then we have the following exact triple

$$(U) \quad 0 \rightarrow B_U \rightarrow A_U \rightarrow J \rightarrow 0.$$

Now let V be any neighborhood of the point x contained in U . As was noted above, there is a homomorphism of exact sequences (*), corresponding to the inclusion $V \subset U$, and commuting with the mappings inside the exact sequences. Hence the existence of a homomorphism of the triple (V) into (U) follows immediately (the module J is mapped identically).

Since, for any neighborhood U of the point x , the module A_U is a submodule in $H^p(X \setminus x, X \setminus U; G)$, and since j is an arbitrary element of the kernel of the

map δ^p , it is enough to prove, obviously, that in passing to the projective limit in the system of exact triples (U) , exactness in the term J is preserved.

From the condition $p_x^p < \infty$ it follows that for every neighborhood U of the point x one can find a neighborhood V such that the image of the module B_V in B_U has a finite number of generators (since the module B_V is contained in the image of the module $H^p(X, X \setminus V; G)$ under the homomorphism

$$H^p(X, X \setminus V; G) \rightarrow H^p(X \setminus x, X \setminus V; G)$$

). Hence it follows at once that, for the same U and V , the image of the module A_V in A_U also has a finite number of generators. Using this circumstance, the remaining part of the proof can be carried out in parallel with the proof of Theorem 5.7 of Chapter 8 in ⁽⁵⁾.

Let N_U^0 be the full inverse image in the group A_U of the element j . If the neighborhood V is contained in U , then, obviously, the image of the set N_V^0 under the mapping $A_V \rightarrow A_U$ is contained in N_U^0 . Consider the family Ψ of all functions $n = \{N_U\}$, defined on the set of all neighborhoods of the point x , whose values are subsets N_U of N_U^0 ; the functions in Ψ must also satisfy the following condition: the image N_V under the mapping $A_V \rightarrow A_U$ is contained in N_U . The family Ψ is nonempty, since the function $n^0 = \{N_U^0\}$ belongs to Ψ . The set Ψ can be partially ordered: $m < n$ if $M_U \subset N_U$ for all U . By Zorn's lemma, Ψ contains a certain maximal linearly ordered subset Ψ' . We shall show that Ψ' has a minimal element. Indeed, let us consider, for some neighborhood U , the set N_U^1 , which is the intersection of all N_U for $n \in \Psi'$. Let V be a neighborhood of the point x , contained in U , and such that the image of the module A_V in A_U has a finite number of generators. For every $n \in \Psi'$, denote by $\tilde{N}_V$ the image of the set N_V in A_U . Since the sets $\tilde{N}_V$ are nonempty and $\tilde{M}_V \subset \tilde{N}_V$ for $m < n$, their intersection over all $n \in \Psi'$ is nonempty (we use the linear compactness of finitely generated modules over a ring with the minimum condition). Since, moreover, $\tilde{N}_V \subset N_U$, the intersection of all N_U (i.e. N_U^1) is also nonempty.

Thus, the sets N_U^1 define a certain function in Ψ (since, as is not hard to verify, the image of every N_V^1 under the mapping $A_V \rightarrow A_U$ is contained in N_U^1). Denote this function by n^1 . Since, obviously, $n^1 < n$ for every $n \in \Psi'$, and since Ψ' is a maximal linearly ordered subset in Ψ , it follows that $n^1 \in \Psi'$. The function n^1 has the following property: if $m < n^1$, $m \in \Psi$, then $m = n^1$.

It remains to show that each N_U^1 consists of a single point (for in this case the function n^1 defines an element of the projective limit of the modules A_U , mapping to $j \in J$). Suppose the contrary, i.e. that for some neighborhood U of the point x the set N_U^1 consists of more than one point. Let a be any point of N_U^1 . Define a new function $m \in \Psi$ as follows: for any neighborhood V of the point x choose a neighborhood W contained in the intersection $V \cap U$; as M_V take the intersection with N_V^1 of the image, under the homomorphism $A_W \rightarrow A_V$, of the set in A_W which is the full inverse image of the point a under

the homomorphism $A_W \rightarrow A_U$. Obviously, $m < n^1$. At the same time $m \neq n^1$, since $M_U = a$. This contradicts the maximality of the set Ψ' . The theorem is proved.

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Received
25 XI 1966

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Note: Figure translations are in progress. See original paper for figures.

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